# THE SOLUTION OF BOEN'S PROBLEM

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A finite $p$-group $P$ is said to be $p$-automorphic if and only if it admits a group of automorphisms $G$ which transitively permutes its elements of order $p$. A standing problem has been the proof of

$C_1$. *$p$-automorphic $p$-groups of odd order are abelian.*

A number of authors have proved special cases of $C_1$ as well as special cases of more general propositions [1, 2, 3, 5, 6, 7, 8]. Both $C_1$ and all of the generalizations of it which have been considered in the literature follow from Theorem 1 which appears below.

In [2] it is observed that if $P$ is a smallest counterexample to $C_1$, then there is associated with $P$, an anticommutative (not necessarily associative) algebra $A$ over GF($p$), whose dimension coincides with the number of elements in a minimal generating set of the $p$-automorphic group $P$. Further, if $G$ is the hypothesized group of automorphisms of $P$, then $G$ also acts as a group of automorphisms of $A$ in such manner that both $A$ and the Frattini-factor group of $P$ are isomorphic as GF($p$)$G$-modules. Accordingly, Kostrikin [6] has introduced the notion of *homogeneous algebra*, i.e. a finite dimensional algebra $A$ over a finite field GF($q$), which admits a group of automorphisms $G$, transitively permuting its nonzero elements. Such algebras enjoy two basic properties: ($P_1$) if $q$ is odd, they are anticommutative [6], and ($P_2$) left multiplication by an element induces a nilpotent transformation of $A$ [2]. Then $C_1$ is a consequence of the proposition:

$C_2$. *If $A$ is an homogeneous algebra of odd characteristic then* $A^2=0$.

One may also define semi-$p$-automorphic $p$-groups (spa-groups) as finite $p$-groups admitting a group of automorphisms $G$ which is transitive on the cyclic subgroups of order $p$. This carries with it the corresponding notion of *spa-algebra*, i.e. an anticommutative finite dimensional algebra $A$ over GF($q$), admitting a group of automorphisms $G$ transitive on the 1-dimensional subspaces of $A$. (Property $P_2$ holds for such an algebra, but $P_1$ must be hypothesized if $q$ is exceeded by the dimension of $A$.) The following two conjectures have been considered in [3, 7, 8]:

$C_3$. *Semi-p-automorphic p-groups of odd order are abelian.*

$C_4$. *If A is a spa-algebra of odd characteristic, then* $A^2=0$.

The following implications hold: $C_4 \Rightarrow C_3 \Rightarrow C_1$, $C_4 \Rightarrow C_2 \Rightarrow C_1$. All of these, however, are consequences of the following

THEOREM 1. *Let A be a finite dimensional algebra over* GF$(q)$ *and suppose G is a group of automorphisms of A which acts transitively on the* 1*-dimensional subspaces of A. Suppose also that* GF$(q)$ *contains more than two elements and that A has dimension greater than one. Then* $A^2=0$ *or A has no zero divisors.*

The theorem differs from $C_4$ in that no hypothesis on anticommutativity is required, and that the result accommodates algebras over fields of characteristic 2.

In the discussion which follows, $n$ will denote either the rank of a $p$-group, or else the dimension of the pertinent algebra. Similarly, $G$ will denote the group of automorphisms (of a $p$-group or algebra) which satisfies the relevant transitivity condition. An easy result is that $C_1$ holds if $G$ is cyclic [5]. In [1] and [2], $C_1$ is proved subject to the condition that either $n \leqq 5$ or that $n \neq 6$ and $p > n^{3n^2}$. This result was greatly improved by Kostrikin [6], who proved that $C_2$ holds if $q > n-6$. Recently in [3], Dornhoff was able to sharpen this to $2q > n-3$.

Nearly two years ago, the author was able to show $C_4$ if either (i) $n$ is a prime, or (ii) $G$ is $p$-solvable, where $p$ is the characteristic of the ground field [8].[1] (The result for the condition (ii) was recently independently proved by D. Passman [7].) The fact that $C_4$ is implied by the $p$-solvability of $G$ seems to be more useful than the information quoted in the previous paragraph. As an easy application of this, we have that a finite group containing one conjugate class of subgroups of order $p$ ($p$ odd) has abelian $p$-Sylow subgroups $S$ if and only if elements of order $p$ in $S$ lie in the center of $S$ (a result which figures in [4]). Moreover, Dornhoff was able to utilize this to show that $C_4$ (as well as $C_3$) is a consequence of $2q > n-3$ (see the final section of [3]).

Theorem 1 is an easy consequence of the following more general theorem whose proof from first principles will appear elsewhere [9].

THEOREM 2. *Let A be a (not necessarily associative) finite dimensional algebra over* GF$(q)$ *where* $q > 2$. *Let B be a left ideal of A satisfying*

[1] These results were submitted to Pacific J. Math. in February and April of 1966 and, to the author's knowledge, still remain there, unrefereed.

$B^2 = 0$. *We suppose that for any* $a \in A$, *left multiplication of* $A$ *by* $a$ *induces a linear transformation of* $A$ *whose restriction to the subspace* $B$ *is nilpotent. Suppose also that* $A$ *admits a group of automorphisms which leaves* $B$ *invariant and transitively permutes the* 1-*dimensional subspaces of* $B$. *Then* $AB = 0$.

## References


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