ON SOME FIXED POINTS THEOREMS IN GENERALIZED COMPLETE METRIC SPACES

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In Theorem 2 of [1][2] A. F. Monna generalized a result by W. A. J. Luxemburg on fixed points [2], valid for one operator in a generalized complete metric space, to a suitable family of operators; this result was later completed by M. Edelstein [3].

Clearly, when the family reduces to a unique element (i.e. $T_i \equiv T$ for all i), one gets Luxemburg's result. But if one considers the family of iterates of T: $T_i = T^i$ $(i = 1, 2, \cdots)$, since Hypothesis 1 of Monna's Theorem requires $d(T_i x, T_i y) \leqq \rho d(x, y)$ $(i = 1, 2, \cdots)$ when $d(x, y) \leqq C$, we must have, in particular, $d(Tx, Ty) \leqq \rho d(x, y)$, and Luxemburg's Theorem applies, providing even with a stronger conclusion than Monna's for this particular situation. In order to include this case as a strict generalization of Luxemburg's result, we relax Hypothesis 1 slightly, thus including also Monna's Theorem, and at the same time we get for the family $\{T^i\}$ a nontrivial result. This last assertion will be clarified with an example. This constitutes §1 of our paper.

In §2 we give some fixed point results for a family of operators with $\rho = 1$.

1. THEOREM 1. *Let (X, d) be a generalized complete metric space,[3] and $\{T_i\}_{i=1,2,\cdots}$ a family of self-mappings of X, closed under composition, such that*

(1) *There exist constants $C > 0$, $0 \leqq \rho < 1$, and an integer $m \geqq 1$ such that if $x, y \in X$ and $d(x, y) \leqq C$, then*

$$d(T_{m+k}x, T_{m+k}y) \leqq \rho d(x, y); \qquad k = 0, 1, 2, \cdots.$$

(2) $T_i = T_j = T_j T_i$; $i, j = 1, 2, \cdots$.

(3) *Let $x_0 \in X$ be arbitrary, and define $x_n = T_n x_{n-1}$ $(n = 1, 2, \cdots)$. Then there exists $N(x_0)$ such that $d(T_{n+k}x_n, x_n) \leqq C$, for $n \geqq N$, $k = 1, 2 \cdots$.*

Then, there exists a $\xi \in X$ such that $x_n \to \xi$ and $T_n \xi \to \xi$ as $n \to \infty$.

Furthermore, (if)

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[2] Numbers in [ ] correspond to References.

[3] We follow Luxemburg's denomination.

(4) $T_n x \to x$, $T_n y \to y$ as $n \to \infty \Rightarrow d(x, y) \leqq C$ *then* ξ *is unique, and* $T_{h+k}\xi = \xi$ $(h = \max(m, N); k = 0, 1, \cdots)$.

(In each case, convergence refers to convergence in the metric d.)

REMARKS. (a) When $m = 1$, we have Monna's Theorem, with Edelstein's completion of it.

(b) Example 1 will show that our Hypothesis 1 is strictly more general than Monna's.

(c) If $T_i = \tilde{T}^i$ $(i = 1, 2, \cdots)$ the conclusion means that ξ is periodic under $\tilde{T}$ with period not greater than $\max(m, N)$.

(d) Since ξ is unique, and $T_k\xi = T_k T_h \xi = T_h T_k \xi$, we have $T_k \xi = \xi$, and ξ is actually a fixed point of all T_k $(k = 1, 2, \cdots)$.

PROOF OF THEOREM 1. It will not be given in detail, since it follows mainly the procedures used by Luxemburg and Monna.

Let $x_0 \in X$. Then, in the usual manner, it can be shown that for $n \geqq N + m$: $d(x_{n+h}, x_n) \leqq \rho^{n-N-m} C(1-\rho)^{-1}$, and all results follow like in Luxemburg, Monna and Edelstein.

We observe, letting $h \to \infty$, that the last inequality gives an estimate of the rate of convergence of the sequence $\{x_n\}$, namely

$$d(x_n, \xi) \leqq \rho^{n-N-m} C(1 - \rho)^{-1}.$$

EXAMPLE 1. Let $X = \{x = e^{i\theta}/0 \leqq \theta \leqq \pi/2\}$ and, for $x, y \in X$, define

$$\begin{aligned} d(x, y) &= 0, \qquad \text{if } \theta_x = \theta_y = \pi/2, \\ &= \left| \operatorname{tg} \theta_x - \operatorname{tg} \theta_y \right|, \qquad \text{otherwise.} \end{aligned}$$

Clearly, $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$, $d(x, y) \leqq d(x, z) + d(y, z)$, for $x, y, z \in X$.

Assume $\{x_n = e^{i\theta_n}\}$ is a Cauchy sequence in (X, d). Since, for $m, n \geqq n_0$, $d(x_m, x_n)$ is bounded, we have that either $\theta_n = \theta = \pi/2$ for $n \geqq n_0$, or $0 \leqq \theta_n < \pi/2 - \epsilon$ for some $\epsilon > 0$, $n \geqq n_0$. In the last case, since

$$\begin{aligned} d(x_m, x_n) &= \left| \operatorname{tg} \theta_n - \operatorname{tg} \theta_m \right| = \left| \operatorname{tg} (\theta_n - \theta_m) \right| \left| 1 + \operatorname{tg} \theta_n \cdot \operatorname{tg} \theta_m \right| \\ &\geqq \left| \operatorname{tg} (\theta_n - \theta_m) \right|, \end{aligned}$$

we have that $\{\theta_n\}$ is a Cauchy sequence in $[0, \pi/2 - \epsilon]$; hence, $\theta_n \to \theta$ as $n \to \infty$. Let $x = e^{i\theta}$. In any case, $d(x_n, x) \to 0$ as $n \to \infty$, and (X, d) is a generalized complete metric space.

Define $T: X \to X$ as follows

$$\begin{aligned} Tx &= e^{i0}, & 0 &\leqq \theta < \pi/8, \\ &= e^{i(\theta + \pi/8)}, & \pi/8 &\leqq \theta < 3\pi/8, \\ &= e^{i(\theta - 3\pi/8)}, & 3\pi/8 &\leqq \theta < \pi/2, \\ &= e^{i0}, & \theta &= \pi/2. \end{aligned}$$

Then

$$\begin{aligned} T^2x &= e^{i0}, & 0 \leqq \theta < \pi/8, \\ &= e^{i(\theta+\pi/4)}, & \pi/8 \leqq \theta < \pi/4, \\ &= e^{i(\theta-\pi/4)}, & \pi/4 \leqq \theta < 3\pi/8, \\ &= e^{i0}, & 3\pi/8 \leqq \theta \leqq \pi/2. \end{aligned}$$

$$\begin{aligned} T^3x &= e^{i\theta}, & 0 \leqq \theta < \pi/8, \\ &= e^{i(\theta-\pi/8)}, & \pi/8 \leqq \theta < \pi/4, \\ &= e^{i0}, & \pi/4 \leqq \theta \leqq \pi/2, \end{aligned}$$

$$T^4x = e^{i0} \qquad 0 \leqq \theta \leqq \pi/2.$$

Hence, neither T nor T^2 nor T^3 satisfy Hypothesis 1 of Theorem 1, for they are not continuous. Obviously, T^4 satisfies this condition with arbitrary $C>0$ and arbitrary ρ, $0 \leqq \rho < 1$, so that in this case $m=4$.

As for Hypothesis 3 of the Theorem, let $x_0 = e^{i\theta_0}$. It is not difficult to see that if, for instance, $C=1$, and:

(a) $0 \leqq \theta_0 < \pi/8$, then $N(x_0)=0$.

(b) $\pi/8 \leqq \theta_0 < 3\pi/8$, then $N(x_0)=2$.

(c) $3\pi/8 \leqq \theta_0 \leqq \pi/2$, then $N(x_0)=1$.

Condition 4 clearly holds.

Also, Remark (d) implies that $Tx_0 = x_0$ iff $x_0 = e^{i\theta}$.

2. Our aim now is to obtain similar results for a family of operators satisfying Hypothesis 1 of Theorem 1 with a strict inequality sign and $\rho=1$. We will have to add some further requirements, for even with $T_i \equiv T$ the theorem would not hold. (See, for instance, [**4**, Remark 2].)

THEOREM 2. *Let (X, d) be a generalized complete metric space, and $\{T_i\}_{i=1,2,\cdots}$ a family of self-mappings of X, such that*

(1) *There exist a constant $C>0$ and an integer $m \geqq 1$ such that if $x, y \in X$ and $0 < d(x,y) \leqq C$ then $d(T_{m+k}x, T_{m+k}y) < d(x,y)$; $k=0, 1, \cdots$.*

(2) *$T_iT_j = T_jT_i$; $i, j = 1, 2, \cdots$.*

(3) *Let $x_0 \in X$ be arbitrary, and define $x_n = T_n x_{n-1}$ $(n=1, 2, \cdots)$. Then there exists $N(x_0)$ such that*

(a) *$d(T_{n+k}x_n, x_n) \leqq C$; $n \geqq N$, $k = 1, 2, \cdots$,*

(b) $$\frac{d(T_{n+k+1}x_{n+i+2}, x_{n+i+2})}{d(T_{n+k+1}x_{n+i+1}, x_{n+i+1})} \leqq \frac{d(T_{n+k+1}x_{n+i+1}, x_{n+i+1})}{d(T_{n+k+1}x_{n+i}, x_{n+i})},$$

whenever the denominators do not vanish and $n \geqq N$, $k=1, 2, \cdots$; $0 \leqq i \leqq k-1$.

$$\text{(c)}\quad \frac{d(T_{n+k+1}x_{n+1}, x_{n+1})}{d(T_{n+k+1}x_n, x_n)} \leqq \frac{d(T_{n+k}x_{n+1}, x_{n+1})}{d(T_{n+k}x_n, x_n)}; \qquad k = 1, 2, \cdots,$$

whenever the denominators do not vanish.

(4) $T_p x = x$, $T_p y = y$, $p \geqq m \Rightarrow d(x, y) \leqq C$.

Then, there exists a $\xi \in X$ *such that* $x_n \to \xi$, *and* $T_n \xi \to \xi$ *as* $n \to \infty$.

Furthermore, if

(5) *For some integer* $h \geqq m$: $T_h T_i = T_{h+i}$ $(i=1, 2, \cdots)$ (*and hence* $\{T_i\}$ *is a finitely generated semigroup*), *then* ξ *is unique, and* $T_i \xi = \xi$; $i=1, 2, \cdots$.

(In each case, convergence refers to convergence in the metric d.)

REMARKS. (a) When $T_i \equiv T$, condition (3c) is obviously satisfied. Moreover, in this case, although (5) implies the trivial transformation $Tx = \xi$, $\forall x \in X$, we get the whole conclusion without it.

(b) When $T_i = \tilde{T}^i$, (5) holds for any integer h, and hence ξ is fixed under $\tilde{T}$.

Before proving the Theorem, we need the following

LEMMA. *If for some integer* $h \geqq m$, *and some integer* p, *we have* $T_h x_p = x_p$, *then* $x_p = x_{p+1} = x_{p+2} = \cdots = \xi$. *Furthermore,* $T_{h+k}\xi = \xi$ *for* $k \geqq K(p, h)$.

PROOF. Let $T_h x_p = x_p$. Then $T_h T_{p+1} x_p = T_{p+1} T_h x_p = T_{p+1} x_p$, i.e. $T_h x_{p+1} = x_{p+1}$. Assume $T_h x_{p+j} = x_{p+j}$ $(j > 1)$. Then

$$T_h x_{p+j+1} = T_h T_{p+j+1} x_{p+j} = T_{p+j+1} T_h x_{p+j} = T_{p+j+1} x_{p+j} = x_{p+j+1},$$

i.e. $T_h x_{p+q} = x_{p+q}$; $q = 0, 1, 2, \cdots$.

Hence $d(x_{p+q}, x_{p+q'}) = d(T_h x_{p+q}, T_h x_{p+q'})$, contradicting (1), unless $x_{p+q} = x_{p+q'}$; for, by (4), $d(x_{p+q}, x_{p+q'}) \leqq C$. Therefore, all elements in the set $\{x_{p+q}\}_{q=0,1,\ldots}$ coincide. Call their common value ξ. Choose K so large that $h+K = p+1$ (in case $h > p$, take $K=0$). Then, if $k \geqq K$: $T_{h+k}\xi = T_{h+k} x_{h+k-1} = x_{h+k} = \xi$, and our Lemma is proved.

PROOF OF THE THEOREM. We will assume that no element of the set $\{x_n\}_{n \geqq m}$ is fixed under any of the elements of $\{T_k\}_{k \geqq m}$. Otherwise, by the Lemma, the first part of the theorem would be already proved (and even the second one, if $h=1$ and $K=0$).

Define $\rho_i(x, y) = d(T_i x, T_i y)/d(x, y)$, for $0 < d(x, y) \leqq C$. Let N be the index specified in (3). Without loss, we may assume $N \geqq m$. (If not, we take a suitable $N' > N$.)

Then $d(x_{N+1}, x_N) \leqq C$,

$$\begin{aligned} d(x_{N+2}, x_{N+1}) &= d(T_{N+2}x_{N+1}, x_{N+1}) = d(T_{N+2}T_{N+1}x_N, T_{N+1}x_N) \\ &= d(T_{N+1}T_{N+2}x_N, T_{N+1}x_N) \\ &= \rho_{N+1}(T_{N+2}x_N, x_N)d(T_{N+2}x_N, x_N) \\ &\leqq C\rho_{N+1}(T_{N+2}x_N, x_N), \end{aligned}$$

where we used (2), (3a) and the definition of $\rho_i(x, y)$.

Using the same arguments

$$\begin{aligned} d(x_{N+3}, x_{N+2}) &= d(T_{N+3}x_{N+2}, x_{N+2}) = d(T_{N+3}T_{N+2}x_{N+1}, T_{N+2}x_{N+1}) \\ &= d(T_{N+2}T_{N+3}x_{N+1}, T_{N+2}x_{N+1}) \\ &= \rho_{N+2}(T_{N+3}x_{N+1}, x_{N+1})d(T_{N+3}x_{N+1}, x_{N+1}) \\ &= \rho_{N+2}(T_{N+3}x_{N+1}, x_{N+1})d(T_{N+3}T_{N+1}x_N, T_{N+1}x_N) \\ &= \rho_{N+2}(T_{N+3}x_{N+1}, x_{N+1})d(T_{N+1}T_{N+3}x_N, T_{N+1}x_N) \\ &= \rho_{N+2}(T_{N+3}x_{N+1}, x_{N+1})\rho_{N+1}(T_{N+3}x_N, x_N)d(T_{N+3}x_N, x_N) \\ &\leqq C\rho_{N+2}(T_{N+3}x_{N+1}, x_{N+1})\rho_{N+1}(T_{N+3}x_N, x_N). \end{aligned}$$

And, in general

$$\begin{aligned} d(x_{N+k+1}, x_{N+k}) &\leqq C\prod_{i=0}^{k-1}\rho_{N+i+1}(T_{N+k+1}x_{N+i}, x_{N+i}) \\ &\leqq C\prod_{i=0}^{k-1}\rho_{N+1}(T_{N+k+1}x_N, x_N) \\ &= C[\rho_{N+1}(T_{N+k+1}x_N, x_N)]^k \\ &\leqq C[\rho_{N+1}(T_{N+1}x_N, x_N)]^k \end{aligned}$$

where we used (3b) and (3c). (We note that $\rho_i(x, y)$ is well defined for all elements in the sequence, because of the assumption made at the beginning of the proof.)

Now, let $n \geqq N$, say $n = N+q$. Then

$$(\#) \qquad \begin{aligned} d(x_{n+p}, x_n) &\leqq \sum_{k=0}^{p-1} d(x_{n+k+1}, x_{n+k}) = \sum_{k=0}^{p-1} d(x_{N+q+k+1}, x_{N+q+k}) \\ &\leqq \sum_{k=0}^{p-1} C[\rho_{N+1}(T_{N+1}x_N, x_N)]^{q+k} \\ &\leqq C[\rho_{N+1}(T_{N+1}x_N, x_N)]^{n-N}/(1-\rho_{N+1}(T_{N+1}x_N, x_N)) \end{aligned}$$

for, by definition, $0 \leqq \rho_i(x, y) < 1$, $(i \geqq m)$.

Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$. Also

$$\begin{aligned} d(T_n\xi, \xi) &\leqq d(T_n\xi, x_n) + d(x_n, \xi) \\ &= d(T_n\xi, T_n x_{n-1}) + d(x_n, \xi) \\ &< d(\xi, x_{n-1}) + d(x_n, \xi), \end{aligned}$$

since for $n > \nu$, $d(x_{n-1}, \xi) \leqq C$.

Therefore, $T_n\xi \to \xi$ as $n \to \infty$, and we have the first part of our theorem.

From (#), on making $p \to \infty$, we get an estimate of the rate of convergence of $\{x_n\}$, namely

$$d(x_n, \xi) \leqq C[\rho_{N+1}(T_{N+1}x_N, x_N)]^{n-N}/(1 - \rho_{N+1}(T_{N+1}x_N, x_N)).$$

Assume now (5) holds. Hence $d(T_hT_i\xi, T_i\xi) = d(T_{h+i}\xi, T_i\xi)$. If we let $i \to \infty$, since T_h is continuous, the first member tends to $d(T_h\xi, \xi)$, while the second one tends to zero. Hence, $T_h\xi = \xi$.

Assume $T_i\eta \to \eta$ as $i \to \infty$. Then, as before, $T_h\eta = \eta$. Furthermore, by (4) $d(\xi, \eta) \leqq C$. If $\xi \neq \eta$, we would have $d(\xi, \eta) = d(T_h\xi, T_h\eta)$, contradicting (1), for $h \geqq m$. Hence, $d(\xi, \eta) = 0$ and ξ is unique.

Since $T_i\xi = T_iT_h\xi = T_hT_i\xi$, and ξ is a unique fixed point of T_h, we have $T_i\xi = \xi$ $(i = 1, 2, \cdots)$ concluding thus the proof of our theorem.

Finally, we give three examples to illustrate the extents and limitations of our results. The first one is a direct application of Theorem 2, for which no previous results can be used. The second one shows that condition (3b) is by no means necessary. To counterballast this, the last example shows a transformation without fixed points, which does not fulfill requirement (3b).

EXAMPLE 2. Let $X = \{x = e^{i\theta}/0 \leqq \theta \leqq 3\pi/2\}$, $d(x,y) = |x - y| = 2|\sin((\theta_x - \theta_y)/2)|$.

Define $T: X \to X$ such that $Tx = e^{i\theta_x/3}$ (i.e. $T_i \equiv T$, all i). Let $0 < C < 2^{1/2}$. Hence, it is clear that $d(Tx, Ty) < d(x, y)$ for $0 < d(x, y) \leqq C$. (The inequality can not be improved, for $d(Tx, Ty)/d(x, y) \to 1$ when $x \to e^{i0}$, $y \to e^{i3\pi/2}$.)

Let $x_0 = e^{i\theta} \in X$, arbitrary. Then, $x_n = Tx_{n-1} = e^{i\theta/3^n} = e^{i\theta_n}$. Since $d(x_{n+1}, x_n) = 2|\sin((\theta_{n+1} - \theta_n)/2)| = 2\,|\sin(\theta/3^{n+1})|$, it is clear that for $n \geqq N_1(x_0)$, $d(x_{n+1}, x_n) \leqq C$, and we have (3a).

In order to show (3b), we observe that in this case the condition reduces to

$$d(x_{n+3}, x_{n+2})/d(x_{n+2}, x_{n+1}) \leqq d(x_{n+2}, x_{n+1})/d(x_{n+1}, x_n)$$

when $n \geqq N(x_0)$ and the denominators do not vanish. This last alternative occurs only when $\theta = 0$, where the final result is trivial. So that we may assume $\theta \neq 0$. Let $\phi = \theta/3^{n+3}$. With this notation, (3b) will hold if and only if

$$2 \sin \phi/2 \sin 3\phi \leqq 2 \sin 3\phi/2 \sin 9\phi.$$

Clearly, taking $n \geqq N_2(x_0)$, we will have $0 \leqq 9\phi \leqq \pi/2$.

Consider the function $f(t) = \sin 3t/\sin t = 2\cos^2 t + \cos 2t$, $0 < t \leqq \pi/2$. A simple computation shows $f'(t) < 0$. Hence, $f(t)$ decreases, and $\sin t/\sin 3t$ increases with t. In particular $\sin \phi/\sin 3\phi < \sin 3\phi/\sin 9\phi$, which is precisely what we wanted to prove.

As stated in Remark (a), (3c) is obviously satisfied.

Also, condition (4) is immediate.

Hence, we can apply our Theorem and conclude: $x_n \to \xi = e^{i0}$ as $n \to \infty$, and $T\xi = \xi$.

We observe that neither Theorem 1 nor Theorem 3 in [5] can be used in this case, whereas Theorem 2 in [5] asserts at best the periodicity of ξ.

EXAMPLE 3. Let X be the nonnegative reals with the Euclidean metric. Define $T: X \to X$ by $Tx = x/(x+1)$. It is easy to show that $d(Tx, Ty) < d(x, y)$ whenever $x \neq y$, so that $C > 0$ can be chosen arbitrarily.

Let $x_0 \in X$, $x_0 \neq 0$, $x_n = Tx_{n-1}$. (The case $x_0 = 0$ is trivial.) Hence $x_n = x_0(nx_0 + 1)^{-1}$ and

$$d(x_{n+1}, x_n) = x_0^2/[(n+1)x_0 + 1](nx_0 + 1).$$

Therefore

$$\frac{d(x_{n+3}, x_{n+2})}{d(x_{n+2}, x_{n+1})} - \frac{d(x_{n+2}, x_{n+1})}{d(x_{n+1}, x_n)} = \frac{2x_0^2}{[(n+3)x_0 + 1][(n+2)x_0 + 1]} > 0$$

and hence (3b) is not satisfied. Nevertheless, $x_n \to 0 \in X$ as $n \to \infty$, $T0 = 0$, and 0 is the only element with this property.

EXAMPLE 4. We will use Rakotch's example mentioned at the beginning of this Section.

Let X be the nonnegative reals with the Euclidean metric, and $Tx = \ln(1 + e^x)$. As shown there $d(Tx, Ty) < d(x, y)$ whenever $x \neq y$, so that $C > 0$ can be chosen arbitrarily. Let $x_0 \in X$, $x_0 \neq 0$, and $x_n = Tx_{n-1}$. (The case $x_0 = 0$ is trivial.) Hence, as shown by induction $x_n = \ln(n + e^{x_0})$. Since $d(x_{n+1}, x_n) = \ln(1 + 1/(n + e^{x_0}))$, we see that for $n \geqq N(x_0)$, $d(x_{n+1}, x_n) \leqq C$. Let $f(t) = (\ln((t+2)/(t+1))/(\ln(t+1)/t))$, $t > 1$. Clearly, $f(t) < 1$, and $f(t) \to 1$ as $t \to \infty$.

Also,

$$\begin{aligned} f'(t) = {} & [(t+1)\ln^2((t+1)/t)]^{-1} \\ & \cdot [(1/t)\ln((t+2)/(t+1)) - (1/(t+2))\ln((t+1)/t)]. \end{aligned}$$

Assume that for some $s>1, f'(s)=0$. Hence, we have the following chain of equivalences.

$$f'(s) = 0 \Leftrightarrow \ln((s+2)/(s+1))^{s+2}$$
$$= \ln((s+1)/s)^s \Leftrightarrow (1+1/(s+1))^{s+2} = (1+1/s)^s.$$

Therefore $(1+1/s)^s > (1+1/(s+1))^{s+1}$, which is clearly impossible, for the function $(1+1/t)^t$ increases with t. Therefore, $f'(t)$ has constant sign. If it were $f'(t)<0$, then $f(t)$ would be a decreasing function, and since $f(t)\to 1$ when $t\to\infty$, we would have $f(t)\geqq 1$, which is impossible. Therefore, $f(t)$ is increasing. In particular: $f(n+1+e^{x_0}) > f(n+e^{x_0})$, i.e. (3b) does not hold.

It is obvious that $x_n \to \infty \notin X$, $Tx_n \to \infty$ as $n\to\infty$, and furthermore T does not have any fixed points in X.

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