RESTRICTED LIE ALGEBRAS OF BOUNDED TYPE

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Introduction. It is known [13] that a Lie algebra over a modular field has indecomposable representations of arbitrarily high dimensionalities. If, however, the Lie algebra and its representations are required to be restricted (see [6, Chapter 5] for definitions), this need no longer be the case.

A restricted Lie algebra for which the degrees of its (restricted) indecomposable representations are bounded by some constant is said to be of bounded type; one for which this is not the case is said to be of unbounded type.

1. The simple three-dimensional Lie algebra, $A_1$. Let $A_1$ be the split simple three-dimensional Lie algebra over the field $K$ of characteristic $p > 3$. Then $A_1$ has a basis $e, f, h$ with $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$ and with $p$-power mapping given by $e^p = f^p = 0$, $h^p = h$.

There are $p$ inequivalent irreducible (restricted) modules for $A_1$, classified by their highest weight. Let $M_\lambda$, $0 \leq \lambda \leq p - 1$, be the irreducible $A_1$-module with highest weight $\lambda$, so that $[M_\lambda : K] = \lambda + 1$ [5].

Let $U$ be the $u$-algebra [6] of $A_1$ and $U = \sum_{\lambda=0}^{p-1} U_\lambda$ its decomposition into its principal indecomposable modules (p.i.m.). Since $U$ is a symmetric algebra [9] each $U_\lambda$ has a unique top and bottom composition factor, these are isomorphic, and each $M_\lambda$ is isomorphic to the top composition factor of some $U_\lambda$ [2].

If $M$ is an $A_1$-module, denote by $M \sim M_{\lambda_1}, M_{\lambda_2}, \ldots, M_{\lambda_p}$ the fact that the $M_{\lambda_j}$, in the given order, are the composition factors of some composition series for $M$.

THEOREM 1. Let $U(\lambda)$, $0 \leq \lambda \leq p - 1$, be a p.i.m. of $U$ whose top composition factor is isomorphic to $M_\lambda$. Then

(i) $U(p-1) \cong M_{p-1}$ and $[U(p-1) : K] = p$.

(ii) If $\lambda \neq p - 1$, then $U(\lambda) \sim M_\lambda, M_\gamma, M_\gamma, M_\lambda$, where $\lambda + \gamma = p - 2$, and $[U(\lambda) : K] = 2p$.

1 These results are contained in a dissertation submitted to Yale University in 1967, written under the supervision of Professor G. B. Seligman. The research was supported by National Science Foundation grant no. GP-1813.

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The radical $R$ of $U$ is the two-sided ideal generated by the elements $e_{p-1}(h+1)$ and $(h+1)^{p-1}$ [10]. It is not hard to show that $R^2$ is generated, as a two-sided ideal, by $e_{p-1}(h+1)^{p-1}$, and $R^3 = 0$.

**Theorem 2.** Let $M$ be an $A_1$-module for which $MR^3 \neq 0$. Then $M$ has a direct summand isomorphic to $U(\lambda)$ for some $\lambda \not= p-1$.

**Theorem 3.** Let $M$ be an indecomposable $A_1$-module for which $M \supset MR \supset MR^2 = 0$. Then there exists a fixed $\lambda$, $0 \leq \lambda \leq p-2$, such that the socle of $M = MR = \text{the direct sum of copies of } M_\lambda$, and $M/MR$ is the direct sum of copies of $M_\gamma$, where $\lambda + \gamma = p-2$.

**Corollary.** If $M$ is an indecomposable $A_1$-module with exactly two composition factors, then $M \sim M_\lambda$, $M_\gamma$ for some $0 \leq \lambda \leq p-2$ and $\lambda + \gamma = p-2$, so that $[M: K] = p$.

**Theorem 4.** For every positive integer $k$, there exists an indecomposable $A_1$-module $M$ with $[M: K] = k$, so that $A_1$ is of unbounded type.

2. **The classical Lie algebras** [8]. Let $T$ be a finite-dimensional associative algebra (with 1), and let $P$ be a subalgebra (with 1) such that $T$ is a finitely generated left and right $P$-module. If $M$ is a right $P$-module, form $M \otimes_P T$ and define $(\sum m_i \otimes t_i) t = \sum m_i \otimes t_i t$ for $m_i \in M$ and $t, t_i \in T$. With this, $M \otimes_P T$ becomes a $T$-module, denoted by $M^T$. Suppose that for any indecomposable $P$-module $M$, the mapping $m \mapsto m \otimes 1$, $m \in M$, is a $P$-isomorphism of $M$ onto a $P$-direct summand of $M^T$. Then if $P$ has indecomposable representations of arbitrarily high dimensionalities, the same is true of $T$ (see [2], for example).

This is used to prove

**Theorem 5.** Let $L$ be a restricted Lie algebra and $A$ a restricted subalgebra of $L$ such that $L = A \oplus \sum_{j=1}^n B_j$ (as vector spaces), where the $B_j$ are restricted subalgebras with $[A, B_j] \subseteq B_j$. Then if $A$ is of unbounded type, so is $L$.

In particular, if $L$ is a classical Lie algebra one may imbed the algebra $A_1$ in $L$ in such a way that the conditions of Theorem 5 are satisfied.

**Theorem 6.** Let $L$ be a classical Lie algebra over a field of characteristic $p > 3$. Then $L$ is of unbounded type.

Suppose $L$ is a Lie algebra over the field $F$ of characteristic $p > 3$, and that for some extension field $K$ of $F$, the Lie algebra $L_K = L \otimes_P K$ is classical. $L$ is called an $F$-form of the classical Lie algebra $L_K$. 
It is known that $K$ may be assumed to be a finite Galois extension of $F$ [12, Chapter 4]. It follows from the next, more general, theorem that the forms of the classical Lie algebras are all of unbounded type.

**Theorem 7.** Let $T$ be a finite-dimensional associative algebra (with 1) over the modular field $F$. Let $K$ be a finite Galois extension of $F$ such that $T_K = T \otimes_K K$ is of unbounded type. Then $T$ is of unbounded type (over $F$).


**Theorem 8.** Let $W_1$ be the Witt algebra [4]. For $k = 1, 2, \ldots$, there exists an indecomposable (restricted) representation of $W_1$ of dimension $kp$ ($p > 2$ is the characteristic of the base field), so that $W_1$ is of unbounded type.

Methods analogous to those used in the proof of Theorem 6 (where the part played by $A_1$ in that theorem is assumed here by $W_1$) may be used to prove

**Theorem 9.** The Jacobson-Witt algebras $W_n$ [4] and their simple subalgebras $V_{m,n}$ [1] are all of unbounded type.

It seems quite likely that all known simple restricted Lie algebras, i.e., those of classical and Cartan type [7], are of unbounded type.  

4. Nilpotent Lie algebras. Let $L$ be a finite-dimensional restricted Lie algebra over the field $K$ of prime characteristic $p$, and set $L^{(1)} = L$, $L^{(j+1)} = [L, L^{(j)}]$ for $j \geq 1$. If $S$ is a subset of $L$, denote by $S^p$ the linear span of the elements $s^p$, $s \in S$, so that $S^{p^{k+1}} \subseteq (S^p)^p$.

**Definition.** Set $L_1 = L$ and $L_k = L^{(k)} + (L^{(k-1)})^p + (L^{(k-2)})^{p^2} + \cdots + L^{p^{k-1}}$ for $k > 1$. The series $L = L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ is the (restricted) lower central series for $L$, and if $L_k = 0$ for some $k > 1$, $L$ is called nil; if $L^{(k)} = 0$ for some $k > 1$, $L$ is called nilpotent.

The $L_k$ are ideals in $L$, $L_k = L^{(k)} + (L_{k-1})^p$, and if $L$ is nil, then it is also nilpotent (although the converse is not true) and $L^{p^s} = 0$ for $s$ sufficiently large [3]. An element $x \in L$ is nilpotent of index $k$ if $x^{p^{k+1}} \neq 0$ and $x^p = 0$ for some $k > 0$; the subalgebra generated by $x$, i.e. the linear span of $x, x^p, x^{p^2}, \ldots, x^{p^{k-1}}$, is called nil-cyclic and is denoted by $\langle x \rangle$. An element $h \in L$ is called separable if $h \in \langle h^p \rangle$, the subalgebra generated by $h^p$, and a commutative (restricted) Lie algebra consisting of separable elements is called toral.

The $u$-algebra of a nil-cyclic Lie algebra is the group algebra of a

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2 Robert Wilson has shown that these same methods can be used to prove that the remaining algebras of Cartan type are also of unbounded type.
cyclic $p$-group and hence is of bounded type. On the other hand, the $u$-algebra of the direct sum of $n > 1$ nil-cyclic Lie algebras is the group algebra of the direct product of $n > 1$ cyclic $p$-groups, and so is of unbounded type. This is used to prove

**Lemma A.** Let $L$ be a nil Lie algebra over the modular field $K$. Then $L$ is of bounded type if and only if $L$ is nil-cyclic.

**Lemma B.** Let $L = H \oplus \langle x \rangle$ (semidirect) over the algebraically closed field $K$ of characteristic $p > 0$, where $H$ is a toral subalgebra and $\langle x \rangle$ is the nil-cyclic ideal generated by the nilpotent element $x$ of index $k > 0$. Then every indecomposable (restricted) $L$-module has dimension $\leq p^k$ and is a homomorphic image of a p.i.m. of the $u$-algebra of $L$.

**Theorem 10.** Let $L$ be a nilpotent (restricted) Lie algebra over an algebraically closed field of characteristic $p$. Then $L$ is of bounded type if and only if $L = H \oplus \langle x \rangle$, where $H$ is a toral ideal and $\langle x \rangle$ is the nil-cyclic ideal generated by the nilpotent element $x$ (either $H$ or $x$ may be trivial).

**Proof.** Suppose $L$ is of bounded type; we may assume $L$ is not nil. One may take $H = L_k$, where $L_k = L_{k+1}$ and $L_k = (L_k)^p$. If $L/H$ is to be of bounded type, it is nil-cyclic and one then shows that the extension $0 \rightarrow H \rightarrow L \rightarrow L/H \rightarrow 0$ splits. The converse is clear.

5. **Solvable Lie algebras.** Even over an algebraically closed field, a solvable restricted Lie algebra need not satisfy Lie’s theorem, that is, a representation of a solvable Lie algebra need not be simultaneously triangularizable. If, however, $L = H \oplus N$ (semidirect), where $H$ is a toral subalgebra and $N$ is a nil ideal, $L$ is said to be strongly solvable and does satisfy Lie’s theorem. In fact, this property characterizes strongly solvable Lie algebras. Over a perfect field, a solvable Lie algebra $L$ is strongly solvable if and only if $[L, L]$ consists of nilpotent elements, in which case the set of all nilpotent elements of $L$ forms a maximal nil ideal [11].

**Lemma C.** Let $L = H \oplus N$ (semidirect) over an algebraically closed field of characteristic $p \geq 3$, where $H$ is a toral subalgebra and $N$ is a nil ideal with basis $e, f$, where $[e, f] = 0$ and $e^p = f^p = 0$. Then $L$ is of unbounded type.

**Theorem 11.** A strongly solvable restricted Lie algebra over an algebraically closed field of characteristic $p \geq 3$ is of bounded type if and only if its maximal nil ideal is nil-cyclic.

**Proof.** Let $L = H \oplus N$, $H$ a toral subalgebra, and $N$ the maximal
nil ideal of $L$; assume neither $H$ nor $N$ is trivial and that $N$ is not nil-cyclic. If $N' = [N, N] + N^p$, then $N'$ is an ideal of $L$ and $N = N/N'$ is a commutative Lie algebra with trivial $p$-power mapping. Then $L/N' = H \oplus \overline{N}$ (semidirect), where $H \cong H$ and $N$ is the direct sum of one-dimensional $H$-modules. The fact that $N$ is nil but not nil-cyclic implies $\dim \overline{N} \geq 2$, and we may, if necessary, factor $N/N'$ by an ideal of codimension 2 and so assume $\dim \overline{N}$ is exactly 2. From Lemma C it follows that $L/N'$, hence $L$, is of unbounded type. The converse is clear.

A Lie algebra $L$ with center $C$ is said to be centrally strongly solvable if $L/C$ is strongly solvable; it is not hard to see that these algebras are characterized by the nilpotency of $[L, L]$.

**Theorem 12.** A centrally strongly solvable restricted Lie algebra $L$ over the algebraically closed field $K$ of characteristic $p \geq 3$ is of bounded type if and only if either

(i) $L$ is strongly solvable with nil-cyclic maximal nil ideal or (ii) $L = H \oplus K e$ (as vector spaces), where $H$ is a toral subalgebra and $0 \neq [e, H] \subsetneq K e$, $0 \neq e^p \in H$.

**Definition.** Let $L$ be a restricted Lie algebra and $M$ a finite-dimensional $L$-module. $M$ may be considered as a commutative restricted Lie algebra with trivial $p$-power mapping. The split extension of $L$ by $M$ is the restricted Lie algebra $L^M$ whose underlying space is $L \oplus M$, with product and $p$-power mapping determined by the action of $L$ on $M$ (see [6]).

It is clear that $M$ is a nil ideal of $L^M$ and that if $L^M$ is of bounded type, so is $L$. Partial conditions for a converse are included in the following:

**Theorem 13.** Let $L$ be a restricted Lie algebra over the algebraically closed field $K$ of characteristic $p > 3$ such that $L$ is of bounded type and either

(i) toral,

(ii) strongly solvable, or

(iii) centrally strongly solvable.

Let $M \neq 0$ be a finite-dimensional $L$-module. Then the split extension $L^M$ is of bounded type if and only if $L$ is toral and $M$ is 1-dimensional.

**Bibliography**


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