RESTRICTED LIE ALGEBRAS OF BOUNDED TYPE

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Introduction. It is known [13] that a Lie algebra over a modular field has indecomposable representations of arbitrarily high dimensionalities. If, however, the Lie algebra and its representations are required to be restricted (see [6, Chapter 5] for definitions), this need no longer be the case.

A restricted Lie algebra for which the degrees of its (restricted) indecomposable representations are bounded by some constant is said to be of *bounded type*; one for which this is not the case is said to be of *unbounded type*.

1. **The simple three-dimensional Lie algebra,** A_1. Let A_1 be the split simple three-dimensional Lie algebra over the field K of characteristic $p>3$. Then A_1 has a basis e, f, h with $[e, f]=h$, $[e, h]=2e$, $[f, h]=-2f$ and with p-power mapping given by $e^p=f^p=0$, $h^p=h$. There are p inequivalent irreducible (restricted) modules for A_1, classified by their highest weight. Let M_λ, $0\leqq\lambda\leqq p-1$, be the irreducible A_1-module with highest weight λ, so that $[M_\lambda: K]=\lambda+1$ [5].

Let U be the u-algebra [6] of A_1 and $U=\sum_{j=1}^{n}\oplus U_j$ its decomposition into its principal indecomposable modules (p.i.m.). Since U is a symmetric algebra [9] each U_j has a unique top and bottom composition factor, these are isomorphic, and each M_λ is isomorphic to the top composition factor of some U_j [2].

If M is an A_1-module, denote by $M\sim M_{\lambda_1}, M_{\lambda_2}, \cdots, M_{\lambda_s}$ the fact that the M_{λ_j}, in the given order, are the composition factors of some composition series for M.

THEOREM 1. *Let* $U(\lambda)$, $0\leqq\lambda\leqq p-1$, *be a p.i.m. of U whose top composition factor is isomorphic to* M_λ. *Then*

(i) $U(p-1)\cong M_{p-1}$ *and* $[U(p-1): K]=p$.

(ii) *If* $\lambda\neq p-1$, *then* $U(\lambda)\sim M_\lambda, M_\gamma, M_\gamma, M_\lambda$, *where* $\lambda+\gamma=p-2$, *and* $[U(\lambda): K]=2p$.

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The radical R of U is the two-sided ideal generated by the elements $e^{p-1}(h+1)$ and $(h+1)f^{p-1}$ [10]. It is not hard to show that R^2 is generated, as a two-sided ideal, by $e^{p-1}(h+1)f^{p-1}$, and $R^3=0$.

THEOREM 2. *Let M be an A_1-module for which $MR^2\neq 0$. Then M has a direct summand isomorphic to $U(\lambda)$ for some $\lambda\neq p-1$.*

THEOREM 3. *Let M be an indecomposable A_1-module for which $M\supset MR\supset MR^2=0$. Then there exists a fixed λ, $0\leqq\lambda\leqq p-2$, such that the socle of $M=MR=$ the direct sum of copies of M_λ, and M/MR is the direct sum of copies of M_γ, where $\lambda+\gamma=p-2$.*

COROLLARY. *If M is an indecomposable A_1-module with exactly two composition factors, then $M\sim M_\lambda$, M_γ for some $0\leqq\lambda\leqq p-2$ and $\lambda+\gamma=p-2$, so that $[M: K]=p$.*

THEOREM 4. *For every positive integer k, there exists an indecomposable A_1-module M with $[M: K]=k$, so that A_1 is of unbounded type.*

2. **The classical Lie algebras** [8]. Let T be a finite-dimensional associative algebra (with 1), and let P be a subalgebra (with 1) such that T is a finitely generated left and right P-module. If M is a right P-module, form $M\otimes_P T$ and define $(\sum_i m_i\otimes t_i)t=\sum_i m_i\otimes t_i t$ for $m_i\in M$ and t, $t_i\in T$. With this, $M\otimes_P T$ becomes a T-module, denoted by M^T. Suppose that for any indecomposable P-module M, the mapping $m\to m\otimes 1$, $m\in M$, is a P-isomorphism of M onto a P-direct summand of M^T. Then if P has indecomposable representations of arbitrarily high dimensionalities, the same is true of T (see [2], for example).

This is used to prove

THEOREM 5. *Let L be a restricted Lie algebra and A a restricted subalgebra of L such that $L=A\oplus\sum_{j=1}^{n}\oplus B_j$ (as vector spaces), where the B_j are restricted subalgebras with $[A, B_j]\subseteq B_j$. Then if A is of unbounded type, so is L.*

In particular, if L is a classical Lie algebra one may imbed the algebra A_1 in L in such a way that the conditions of Theorem 5 are satisfied.

THEOREM 6. *Let L be a classical Lie algebra over a field of characteristic $p>3$. Then L is of unbounded type.*

Suppose L is a Lie algebra over the field F of characteristic $p>3$, and that for some extension field K of F, the Lie algebra $L_K=L\otimes_F K$ is classical. L is called an *F-form* of the classical Lie algebra L_K.

It is known that K may be assumed to be a finite Galois extension of F [12, Chapter 4]. It follows from the next, more general, theorem that the forms of the classical Lie algebras are all of unbounded type.

THEOREM 7. *Let T be a finite-dimensional associative algebra (with 1) over the modular field F. Let K be a finite Galois extension of F such that $T_K = T\otimes_F K$ is of unbounded type. Then T is of unbounded type (over F).*

3. The Jacobson-Witt algebras and simple subalgebras.

THEOREM 8. *Let W_1 be the Witt algebra [4]. For $k=1, 2, \cdots$, there exists an indecomposable (restricted) representation of W_1 of dimension kp ($p>2$ is the characteristic of the base field), so that W_1 is of unbounded type.*

Methods analogous to those used in the proof of Theorem 6 (where the part played by A_1 in that theorem is assumed here by W_1) may be used to prove

THEOREM 9. *The Jacobson-Witt algebras W_n [4] and their simple subalgebras $V_{m,\mu}$ [1] are all of unbounded type.*

It seems quite likely that all known simple restricted Lie algebras, i.e., those of classical and Cartan type [7], are of unbounded type.[2]

4. Nilpotent Lie algebras. Let L be a finite-dimensional restricted Lie algebra over the field K of prime characteristic p, and set $L^{(1)}=L$, $L^{(j+1)}=[L, L^{(j)}]$ for $j\geqq 1$. If S is a subset of L, denote by S^p the linear span of the elements s^p, $s\in S$, so that $S^{p^{k+1}}\subseteq (S^{p^k})^p$.

DEFINITION. Set $L_1=L$ and $L_k=L^{(k)}+(L^{(k-1)})^p+(L^{(k-2)})^{p^2}+\cdots+L^{p^{k-1}}$ for $k>1$. The series $L=L_1\supseteq L_2\supseteq L_3\supseteq\cdots$ is the (restricted) lower central series for L, and if $L_k=0$ for some $k>1$, L is called *nil*; if $L^{(k)}=0$ for some $k>1$, L is called *nilpotent.*

The L_k are ideals in L, $L_k=L^{(k)}+(L_{k-1})^p$, and if L is nil, then it is also nilpotent (although the converse in not true) and $L^{p^s}=0$ for s sufficiently large [3]. An element $x\in L$ is nilpotent of index k if $x^{p^{k-1}}\neq 0$ and $x^{p^k}=0$ for some $k>0$; the subalgebra generated by x, i.e. the linear span of $x, x^p, x^{p^2}, \cdots, x^{p^{k-1}}$, is called nil-cyclic and is denoted by $\langle x\rangle$. An element $h\in L$ is called separable if $h\in\langle h^p\rangle$, the subalgebra generated by h^p, and a commutative (restricted) Lie algebra consisting of separable elements is called toral.

The u-algebra of a nil-cyclic Lie algebra is the group algebra of a

[2] Robert Wilson has shown that these same methods can be used to prove that the remaining algebras of Cartan type are also of unbounded type.

cyclic p-group and hence is of bounded type. On the other hand, the u-algebra of the direct sum of $n>1$ nil-cyclic Lie algebras is the group algebra of the direct product of $n>1$ cyclic p-groups, and so is of unbounded type. This is used to prove

LEMMA A. *Let L be a nil Lie algebra over the modular field K. Then L is of bounded type if and only if L is nil-cyclic.*

LEMMA B. *Let $L=H\oplus\langle x\rangle$ (semidirect) over the algebraically closed field K of characteristic $p>0$, where H is a toral subalgebra and $\langle x\rangle$ is the nil-cyclic ideal generated by the nilpotent element x of index $k>0$. Then every indecomposable (restricted) L-module has dimension $\leqq p^k$ and is a homomorphic image of a p.i.m. of the u-algebra of L.*

THEOREM 10. *Let L be a nilpotent (restricted) Lie algebra over an algebraically closed field of characteristic p. Then L is of bounded type if and only if $L=H\oplus\langle x\rangle$, where H is a toral ideal and $\langle x\rangle$ is the nil-cyclic ideal generated by the nilpotent element x (either H or x may be trivial).*

PROOF. Suppose L is of bounded type; we may assume L is not nil. One may take $H=L_k$, where $L_k=L_{k+1}$ and $L_k=(L_k)^p$. If L/H is to be of bounded type, it is nil-cyclic and one then shows that the extension $0\to H\to L\to L/H\to 0$ splits. The converse is clear.

5. **Solvable Lie algebras.** Even over an algebraically closed field, a solvable restricted Lie algebra need not satisfy Lie's theorem, that is, a representation of a solvable Lie algebra need not be simultaneously triangularizable. If, however, $L=H\oplus N$ (semidirect), where H is a toral subalgebra and N is a nil ideal, L is said to be *strongly solvable* and does satisfy Lie's theorem. In fact, this property characterizes strongly solvable Lie algebras. Over a perfect field, a solvable Lie algebra L is strongly solvable if and only if $[L, L]$ consists of nilpotent elements, in which case the set of all nilpotent elements of L forms a maximal nil ideal [**11**].

LEMMA C. *Let $L=H\oplus N$ (semidirect) over an algebraically closed field of characteristic $p\geqq 3$, where H is a toral subalgebra and N is a nil ideal with basis e, f, where $[e,f]=0$ and $e^p=f^p=0$. Then L is of unbounded type.*

THEOREM 11. *A strongly solvable restricted Lie algebra over an algebraically closed field of characteristic $p\geqq 3$ is of bounded type if and only if its maximal nil ideal is nil-cyclic.*

PROOF. Let $L=H\oplus N$, H a toral subalgebra, and N the maximal nil ideal of L; assume neither H nor N is trivial and that N is not nil-cyclic. If $N' = [N, N] + N^p$, then N' is an ideal of L and $\overline{N} = N/N'$ is a commutative Lie algebra with trivial p-power mapping. Then $L/N' = \overline{H} \oplus \overline{N}$ (semidirect), where $\overline{H} \cong H$ and $\overline{N}$ is the direct sum of one-dimensional $\overline{H}$-modules. The fact that N is nil but not nil-cyclic implies dim $\overline{N} \geqq 2$, and we may, if necessary, factor N/N' by an ideal of codimension 2 and so assume dim $\overline{N}$ is exactly 2. From Lemma C it follows that L/N', hence L, is of unbounded type. The converse is clear.

A Lie algebra L with center C is said to be *centrally strongly solvable* if L/C is strongly solvable; it is not hard to see that these algebras are characterized by the nilpotency of $[L, L]$.

THEOREM 12. *A centrally strongly solvable restricted Lie algebra L over the algebraically closed field K of characteristic $p \geqq 3$ is of bounded type if and only if either*

(i) *L is strongly solvable with nil-cyclic maximal nil ideal or* (ii) $L = H \oplus Ke$ *(as vector spaces), where H is a toral subalgebra and* $0 \neq [e, H] \subseteq Ke$, $0 \neq e^p \in H$.

DEFINITION. Let L be a restricted Lie algebra and M a finite-dimensional L-module. M may be considered as a commutative restricted Lie algebra with trivial p-power mapping. The *split extension of L by M* is the restricted Lie algebra L^M whose underlying space is $L \oplus M$, with product and p-power mapping determined by the action of L on M (see [6]).

It is clear that M is a nil ideal of L^M and that if L^M is of bounded type, so is L. Partial conditions for a converse are included in the following:

THEOREM 13. *Let L be a restricted Lie algebra over the algebraically closed field K of characteristic $p > 3$ such that L is of bounded type and either*

(i) *toral,*

(ii) *strongly solvable, or*

(iii) *centrally strongly solvable.*

Let $M \neq 0$ be a finite-dimensional L-module. Then the split extension L^M is of bounded type if and only if L is toral and M is 1*-dimensional.*

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