This derivation of Theorem 2 from Theorem 1 was shown to us by C. T. C. Wall.

REFERENCES


UNIVERSITY OF CALIFORNIA, BERKELEY

ON THE NORM OF STABLE MEASURES

BY LUDWIG ARNOLD AND JOHANNES MICHALICEK

Communicated by R. Creighton Buck, October 12, 1967

1. **Limits of convolution powers and stable measures.** Let $M(R)$ denote the Banach algebra of all complex-valued regular finite measures defined on the Borel sets of the real line $R$, where multiplication is defined by convolution, and

   $$||\mu|| = \sup \sum |\mu(R_i)|,$$

   the supremum being taken over all finite collections of pairwise disjoint sets $R$, whose union is $R$. Let $B(R)$ be the set of all Fourier transforms of measures in $M(R)$.

   In [1], we characterized all possible limits

   $$\lim_{n \to \infty} (\rho(t/B_n))^n \exp (iA_n) = \mu(t) \quad \text{for all } t \neq 0,$$

   where $A_n \in \mathcal{R}$, $B_n > 0$, $\varphi$, $\mu \in B(R)$. This is a generalization of an old problem in probability theory (see e.g. [4]). One can show that a measure $\mu$ appears as a limit if and only if it is *stable*, i.e. has the following property: For all $a > 0$, $b > 0$ there exist $c > 0$ and $\gamma \in \mathcal{R}$ such that

   (1) $\mu(at)\mu(bt) = \mu(ct) \exp (i\gamma t) \quad \text{for all } t \in \mathcal{R}$.

---

1 Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No. DA-31-124-ARO-D-462.
In other words, a stable measure convolved with itself reproduces itself after being properly shifted and scaled. Consequently, stable measures may be considered as a substitute for idempotent measures, which except for degenerate ones do not exist on the real line.

Besides the degenerate measure $\mu = 0$ and $\mu = \delta_\beta$ (unit mass at $x = \beta$), a measure is a solution of (1) if and only if its Fourier transform is of the form

$$\hat{\mu}(t) = \exp \left( -c \left| t \right|^\alpha + i\beta t \right) \text{ for } t \geq 0,$$

$$= \exp \left( -d \left| t \right|^\alpha + i\beta t \right) \text{ for } t < 0,$$

or

$$\hat{\mu}(t) = \exp \left( -c \left| t \right|^\alpha + i\beta \log \left| t \right| \right) \text{ for } t \geq 0,$$

$$= \exp \left( -d \left| t \right|^\alpha + i\beta \log \left| t \right| \right) \text{ for } t < 0,$$

where $\beta \in \mathbb{R}, \alpha \in \mathbb{R}, \alpha \neq 0; c$ and $d$ are complex constants with $\text{Re}(c) > 0$, $\text{Re}(d) > 0$. For $\alpha > 0$ the corresponding measure $\mu$ is absolutely continuous; for $\alpha < 0$ the measure $\delta_\beta - \mu$ is absolutely continuous.

2. Symmetric real-valued stable measures. By (1),

$$\|\mu * \mu\| = \|\mu\|$$

for every stable measure. Therefore, either $\|\mu\| = 0$ or $\|\mu\| \geq 1$. The stable probability measures clearly have $\|\mu\| = 1$. One sees easily that $\|\mu\| > 1$ for every stable measure which is not a probability measure. For $\alpha < 0$ we even have $\|\mu\| > 2$.

In this section we confine ourselves to

$$\hat{\mu}_\alpha(t) = \exp \left( -t \left| t \right|^\alpha \right), \quad \alpha \in \mathbb{R}, \alpha \neq 0.$$

Clearly $\|\mu_\alpha\|$ is equal to 1 for $0 < \alpha \leq 2$, is bigger than 1 for $\alpha > 2$, and is bigger than 2 for $\alpha < 0$.

Our tool will be an approximation of $\hat{\mu}_\alpha$ by a function whose norm can be calculated, and the repeated use of

**Lemma 1** (Beurling [2]), (i) Let $\phi$ be absolutely continuous and $\phi, \phi' \in L_2(\mathbb{R})$. Then $\phi = \hat{\mu} \in B(\mathbb{R})$, $\mu$ is absolutely continuous, and

$$\|\mu\| \leq \left( \int_{-\infty}^{\infty} |\phi(t)|^2 dt \int_{-\infty}^{\infty} |\phi'(t)|^2 dt \right)^{1/4}.$$

(ii) An even function $\phi$ is in $B(\mathbb{R})$ if $\phi(t) \to 0$ ($t \to \infty$) and if the integral below is convergent. Then, putting $\phi = \hat{\mu}$,

$$\|\mu\| \leq \int_{0}^{\infty} t \left| d\phi'(t) \right|.$$
Theorem 1. (i) For $\alpha < 0$,

$$2 < \|\mu_\alpha\| \leq 2 + ((2\alpha(\alpha - 1))^{1/2} - \alpha) \exp(1/\alpha - 1).$$

(ii) If $\alpha \beta > 0$ then

$$\|\mu_\alpha - \mu_\beta\| \leq \|\beta - \alpha\| K(\alpha, \beta),$$

where $K$ is locally bounded.

Proof. (i) The inflection points of $\mu_\alpha$ are $\pm t_0$ where $t_0 = ((\alpha - 1)/\alpha)^{1/\alpha}$. Approximate $1 - \mu_\alpha$ by

$$g_\alpha(t) = 1 - \mu_\alpha(t) \quad \text{for} \quad |t| > t_0,$$

$$= 1 + \mu_\alpha'(-t_0)(t + t_0) - \mu_\alpha(-t_0) \quad \text{for} \quad -t_0 \leq t < 0,$$

$$= 1 + \mu_\alpha'(t_0)(t - t_0) - \mu_\alpha(t_0) \quad \text{for} \quad 0 \leq t \leq t_0.$$

The function $g_\alpha$ is even and concave in $(0, \infty)$, and therefore by Polya's criterion is positive definite. Thus $g_\alpha = \rho_\alpha$ where $\|\nu_\alpha\| = g_\alpha(0) = 1 - \exp(1/\alpha - 1)$. For the remainder $\rho_\alpha - (1 - \mu_\alpha)$, we find by Lemma 1 (i)

$$\|\nu_\alpha - (\delta_0 - \mu_\alpha)\| \leq (2\alpha(\alpha - 1))^{1/2} \exp(1/\alpha - 1).$$

(ii) Lemma 1(ii) yields

$$\|\mu_\alpha - \mu_\beta\| \leq \int_0^\infty t \left| \frac{\partial^2 \nu_\gamma(t)}{\partial \gamma^2} \right| \, dt.$$

By the mean value theorem applied to the variable $\alpha$,\n
$$\rho_\alpha'(t) - \rho_\alpha''(t) = (\beta - \alpha) \left. \frac{\partial^2 \nu_\gamma(t)}{\partial \gamma^2} \right|_{\gamma=\alpha + (\beta-\alpha)\theta},$$

$0 < \theta < 1$. An elementary calculation yields

$$\int_0^\infty t \left| \frac{\partial^2 \nu_\gamma(t)}{\partial \gamma^2} \right| \, dt \leq K(\alpha, \beta),$$

where $K$ is locally bounded.

Corollary 1. (i) $\lim_{\alpha \to 0} \|\mu_\alpha\| = 2$.

(ii) The function $\alpha \to \mu_\alpha$ mapping $R - \{0\}$ into $M(R)$ is continuous with respect to the norm topology in $M(R)$.

It should be mentioned that if we define $\mu_0 = e^{-1}\delta_0$, then $\mu_\alpha$ is continuous at $\alpha = 0$ in the weak* topology of $M(R)$.

3. Asymptotic behavior of $\|\mu_\alpha\|$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
THEOREM 2. As $|\alpha| \to \infty$,
\[ \| \mu_{\alpha} \| = \left( \frac{4}{\pi^2} \right) \log |\alpha| + O(1). \]

For the proof of this fact we need the following

LEMMA 2. Consider the trapezoid-shaped function
\[ \rho_{a,b}(t) = 0 \quad \text{for} \quad t \geq b, \]
\[ = \left( \frac{1}{a} \right) (b - |t|) \quad \text{for} \quad b - a \leq |t| < b, \]
\[ = 1 \quad \text{for} \quad |t| < b - a, \]
where $b > a > 0$. Then, for $b/a \to \infty$,
\[ \| \rho_{a,b} \| = \left( \frac{4}{\pi^2} \right) \log \left( \frac{b}{a} \right) + O(1). \]

For the proof write $\rho_{a,b} = \delta_1 + \delta_2$, where
\[ \delta_1(t) = \sum_{|k| \leq \lfloor b/a \rfloor - 1} \delta(t + ka) \]
and
\[ \delta(t) = 1 - \frac{|t|}{a} \quad \text{for} \quad |t| \leq a, \]
\[ = 0 \quad \text{for} \quad |t| > a. \]

Lemma 1 (i) applied to $\sigma_2 = \rho_{a,b} - \sigma_1$ yields $\| \sigma_2 \| \leq 2$. Furthermore, by direct calculation using Poisson’s summation formula, we obtain
\[ \| \sigma_2 \| = \left( \frac{1}{\pi} \right) \int_{-\pi}^{\pi} \left| D_{\lfloor b/a \rfloor - 1}(x) \right| \, dx, \]
where $D_n$ is the Dirichlet kernel.

PROOF OF THEOREM 2. Assume $\alpha \geq 1$. Approximate $\rho_{\alpha}$ by a trapezoid $\rho_{\alpha}$ so that its sides coincide with the tangents at the inflection points of $\rho_{\alpha}$. This leads to $b/a = \alpha \exp(1/\alpha - 1)$, and by the above lemma for $\alpha \to \infty$,
\[ \| \rho_{\alpha} \| = \left( \frac{4}{\pi^2} \right) \log \alpha + O(1). \]

Again by Lemma 1 (i), $\| \rho_{\alpha} \| - \| \mu_{\alpha} \| = O(1)$. We proceed similarly for $\alpha > 0$.

In the same way we can show that $\| \mu_{\alpha} + \mu_{\alpha} \| = O(1)$. Although $\| \mu_{\alpha} \| \to \infty$ we have for the densities $g_{\alpha}$ of $\mu_{\alpha}$
\[ g_{\alpha}(x) \to \frac{(\sin x)}{\pi x} \quad \text{as} \quad \alpha \to \infty \]
in the norm of $L_2(\mathbb{R})$, as Parseval’s equation shows.
Corollary 2. For every $\epsilon > 0$ there is a $\mu \in M(R)$ such that $||\mu|| = 1$, but $||\mu * \mu|| < \epsilon$.

To see this, choose $\mu_a$ such that $||\mu_a|| > \epsilon^{-1}$. Now take $\mu = \mu_a/||\mu_a||$ and use (2).

Corollary 2 is true also in $M(G)$, where $G$ is the circle group or any compact connected abelian group, since in such a group there exist idempotent measures with arbitrarily large norm. See Cohen [3].

4. A skew case. Consider the stable measures $\mu_{c,a}$ corresponding to

\[
\mu_{c,a}(t) = \exp (-c |t|^a) \quad \text{for } t \geq 0, \\
\quad = \exp (-|t|^a) \quad \text{for } t < 0,
\]

where $a \in (0, 1)$ and $c \in R$.

Theorem 3. For $c \to \infty$,

\[
2 \log c + O(1) \leq ||\mu_{c,a}|| \leq 2 \left| 2\alpha \exp \left( \frac{1}{\alpha} - 1 \right) - 1 \right| \log c + O(1).
\]

A technique similar to the one used in Theorem 2 leads to the conjugate Fejér kernel rather than to the Dirichlet kernel.

References