NEW SIMPLE LIE ALGEBRAS OF TYPE $D_4$

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This brief note is to demonstrate the existence of a new class of (exceptional) Lie algebras of type $D_4$. The construction stems from a cyclic sixth degree extension $P/\Phi$, together with an element $\gamma$ of norm 1 in the unique cubic subfield $F/\Phi$ of $P/\Phi$, where $\gamma \in N_{P/F}(P^*)$. Each such $\gamma$ will determine a non-Jordan (see [1] for definition) Lie algebra $\mathfrak{g}(\gamma)$, of type $D_4$. Two algebras of this form, $\mathfrak{g}(\gamma)$ and $\mathfrak{g}(\rho)$, will be isomorphic if and only if $\gamma$ differs from a conjugate of $\rho$ by a norm in $N_{P/F}(P^*)$. The possibility of obtaining new $D_4$'s from such a construction was first conjectured in [2].

We shall make free use of the well-known theory of finite Galois descent for nonassociative algebras and all the results which we use may be found in ([5, Chapter 10]).

0. Preliminaries. We assume without further mention that all fields which appear here have characteristic unequal to 2 or 3.

Let $\mathfrak{H}$ be a split exceptional central simple Jordan algebra over $P$, \{$e_1, e_2, e_3$\} a set of supplementary orthogonal primitive idempotents and let $\mathfrak{D} = \mathfrak{H}(\mathfrak{Z}, Pe_e)$ be the subalgebra of the derivation algebra of $\mathfrak{H}$ annihilating $\Sigma P e_e.$ Then $\mathfrak{D}$ is the split $D_4$. If $\mathfrak{g}$ is a $\Phi$-algebra form of $\mathfrak{D}(P \supset \Phi)$, then we let $\mathfrak{g}^*$ be the $\Phi$-subalgebra of $\text{End}_P \mathfrak{g}$ generated by $\mathfrak{g}$ (we view $\mathfrak{g}$ as a $\Phi$-subspace of $\mathfrak{D}$ which contains a $\Phi$-basis which is also a $P$-basis for $\mathfrak{D}$). It is known that $(\mathfrak{g}^*)_P \cong P_3 \oplus P_8 \oplus P_8$. $\mathfrak{g}$ is special (i.e., has the form $\mathfrak{g}(\mathfrak{A}, J)$ where $(\mathfrak{A}, J)$ is a central simple associative algebra of degree 8 with involution) if and only if $\mathfrak{g}^*$ has proper ideals. When $\mathfrak{g}^*$ is simple, i.e. when $\mathfrak{g}$ is exceptional, then $\mathfrak{g}$ is of known type—a Jordan $D_4$—if and only if $\mathfrak{g}^*$ is a total matrix algebra over its center. $\mathfrak{g}$ is of type $D_{411}$ ($D_{411}$) if the center of $\mathfrak{g}^*$ is a cyclic (noncyclic) extension of $\Phi$.

If $\mathfrak{g}$ is of type $D_{411}$ and $F$ is the center of $\mathfrak{g}^*$—the canonical $D_{41}$-field extension of $\mathfrak{g}$—then $\mathfrak{g}$ is a non-Jordan $D_{411}$ if and only if none of the simple components of $(\mathfrak{g}^*)_F$ is a total matrix algebra.

We shall need some technical information about the structure of split Cayley algebra. For this we refer to [6] and for convenience list the results below for reference.

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Let $\mathbb{C}$ be the vector space of all $2 \times 2$ matrices.

\[
\begin{pmatrix}
\alpha & a \\
\beta & b
\end{pmatrix}
\]

where $\alpha, \beta \in P$ and $a, b \in P^{(3)} = P \times P \times P$. $\mathbb{C}$ is equipped with a bilinear multiplication and an involution, which are defined by

\[
\begin{pmatrix}
\alpha & a \\
\beta & b
\end{pmatrix}
\begin{pmatrix}
\gamma & c \\
\delta & d
\end{pmatrix} =
\begin{pmatrix}
\alpha \gamma + a \cdot d & \alpha c + \delta a - b \wedge d \\
\gamma b + \beta d + a \wedge c & \beta \delta + b \cdot c
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\alpha & a \\
\beta & b
\end{pmatrix} =
\begin{pmatrix}
\beta & -a \\
-b & \alpha
\end{pmatrix}
\]

where $a \cdot d$ and $b \wedge d$ denote the usual dot and cross product in $P^{(3)}$. $\mathbb{C}$ is a split Cayley algebra over $P$.

The mapping $x \mapsto xS = n(x) \in P \subset \mathbb{C}$ is a nondegenerate quadratic form of maximal Witt index, the generic norm on $\mathbb{C}$. If $\{e_1, e_2, e_3\}$ is the usual cartesian basis for $P^{(3)}$ and we define $u_i \in \mathbb{C}$ by

\[
u_i = \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix}, \quad u_{i+4} = -2 \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix}, \quad 1 \leq i \leq 3
\]

then $u_1, \ldots, u_8$ is a basis for $\mathbb{C}$ where $n(u_i, u_j) = \delta_{i+4, j}$, $n(x, y)$ denoting the norm bilinear form of $n(x)$ with $i + 4$ taken modulo 8.

The multiplication table for $\mathbb{C}$, which is given in [6] for this basis, will be invaluable. (See p. 483.)

The product $u_{i+4}$ is found in the $i$th-row, $j$th-column.

1. **Lemma 1.** Let $\mathbb{C}$ be the split Cayley algebra over $P$ and let $\gamma_1, \gamma_2, \gamma_3 \in P^*$ with $\gamma_1 \gamma_2 \gamma_3 \in (P^*)^3$. Then there exists a related triple $(T_1, T_2, T_3)$ of proper similarities in $\mathbb{C}$ where each $T_i$ is selfadjoint with ratio $\gamma_i$. In particular, $(T_i)^3 = (\gamma_1, \gamma_2, \gamma_3)$.

**Proof.** (See [4] for the definition of related triples.) We take $\mathbb{C}$ as in the preceding section and assume without loss of generality that $\gamma_1 \gamma_2 \gamma_3 = 1$. Let $T_i, i = 1, 2, 3$, be the linear transformation whose matrix $T_i$, with respect to $u_1, \ldots, u_8$, is given on the following page.
Using the fact that \( \{u_1, \cdots, u_4\}, \{u_5, \cdots, u_8\} \) span supplementary totally isotropic subspaces, it is easy to see that \( n(u_i, u_j T_k) = 0 \) if \( i \neq j \). This implies that each \( T_i \) is self adjoint. We have

\[
\begin{align*}
T_1 &= \begin{bmatrix}
0 & \frac{1}{2} \gamma_1 \\
2 & \frac{1}{2} \gamma_1 \\
\end{bmatrix} \\
T_2 &= \begin{bmatrix}
0 & \frac{1}{2} \gamma_2 \\
2 & \frac{1}{2} \gamma_2 \\
\end{bmatrix} \\
T_3 &= \begin{bmatrix}
0 & \frac{1}{2} \gamma_3 \\
2 & \frac{1}{2} \gamma_3 \\
\end{bmatrix}
\end{align*}
\]

Using the fact that \( \{u_1, \cdots, u_4\}, \{u_5, \cdots, u_8\} \) span supplementary totally isotropic subspaces, it is easy to see that \( n(u_i, u_j T_k) = 0 \) if \( i \neq j \). This implies that each \( T_i \) is self adjoint. We have

\[
\begin{align*}
n(u_i T_k, u_j T_k) &= 0 = n(u_i, u_j) \quad \text{if } j \neq i + 4 \pmod{8} \\
n(u_i T_k, u_{i+4} T_k) &= \gamma_k n(u_i, u_{i+4}) \quad 1 \leq i \leq 3 \\
n(u_4 T_k, u_8 T_k) &= -\gamma_2 \gamma_1 \gamma_8 n(u_4, u_8) = \gamma_k n(u_4, u_8) \quad \text{where the}
\end{align*}
\]

subscripts on the \( \gamma \)'s are taken modulo 3.
This shows that $T_k$ is a similarity of ratio $\gamma_k$. To complete the proof we must show that

$$u_i u_j T_1 = \gamma_1(u_i T_2)(u_j T_3)$$

for all $i, j$.

By examining the multiplication table for the $u$'s, we see that $u_i u_j = 0$ implies $(u_i T_2)(u_j T_3) = 0$. The remaining 32 cases are verified by straightforward computations. q.e.d.

**The construction.** Let $P/\Phi$ be a cyclic sixth degree Galois extension with $F/\Phi$ the cubic subfield of $P/\Phi$, and let $s$ be a generator for $\text{gal}(P/\Phi)$. Choose $\gamma \in F^*$ with $1 = N_{F/\Phi}(\gamma) = \gamma^2 \gamma^3$.

Take $C_0$ as the split Cayley algebra over $\Phi$ with basis $\{u_1, \ldots, u_8\}$ as described in §0 and let $C = C_0 [s]$ be the split Cayley algebra over $P$ with basis $\{u_1, \ldots, u_8\}$. We let $T = (T_1, T_2, T_3)$ be the related triple of similarities in $\text{S}$ constructed in Lemma 1 with the ingredients $\gamma, \gamma^3, \gamma^4$. Finally let $S$ be the $s$-linear automorphism of $C$ which fixes $C_0$. Let $\mathfrak{g} = \mathfrak{g}(C_0)$ and let $[\langle 123 \rangle, S]$ and $[1, T]$ be the transformations in $\Gamma L_8(3/\Sigma P e_i)$ as defined in [1, Equations 9 and 5].

**Lemma 2.** $[\langle 123 \rangle, S][1, T] = [1, T][\langle 123 \rangle, S]$.

**Proof.** We must show that $ST_1 = T_2 S, ST_2 = T_3 S$ and $ST_3 = T_1 S$. If we define $(\alpha_i)_* = (\alpha_i')$ in $P$, and if $T_i$ denotes the matrix of $T_i$ with respect to $\{u_1, \ldots, u_8\}$, then our conditions reduce to $T_1 = T_2^*, T_2 = T_3^*$ and $T_3 = T_1^*$. But this is immediate from the form of $T_i$ given in Lemma 1.

Assume now that $\gamma \in F, F(P^*)$ (this assumption is nonvacuous over finite algebraic number fields) and let $C(\gamma)$ be the transformation $[\langle 123 \rangle, S][1, T]$ in $\mathfrak{g}(C_0)$. $C(\gamma)$ is $s$-linear and $C(\gamma)^0 = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = \gamma^2, \alpha_2 = (\gamma^3)\gamma$ and $\alpha_3 = (\gamma^4)\gamma$. It follows from this that conjugation by $C(\gamma)$ induces a pre-cocycle of $\text{gal}(P/\Phi)$ in $\text{Aut}_s \mathfrak{g}, \mathfrak{g} = \mathfrak{g}(3/\Sigma P e_i)$, and hence fixes a $\Phi$-form, say $\mathfrak{g}(\gamma)$. $\mathfrak{g}(\gamma)$ is clearly of type $D_{41}$ with $F/\Phi$ as its canonical $D_{41}$-field extension. Since the division algebra parts of the simple ideals of $\mathfrak{g}(\gamma)_P$ are the cyclic algebras $(P/F, \gamma^2), (P/F, (\gamma^3)^\gamma), (P/F, (\gamma^4)^\gamma)$ and $\gamma \in F, F(P^*)$, we see that $\mathfrak{g}(\gamma)$ is a non-Jordan $D_{41}$. Observe that the algebra $\mathfrak{g}(\gamma)$ is a twist of a Steinberg $D_{41}$ and that this is precisely the situation conjectured at the end of [2].

**Isomorphism conditions.** Let $P/\Phi$ be as above. For any $\gamma \in F^*$ with $\gamma \gamma^* \gamma^8 = 1$, we can define the algebra $\mathfrak{g}(\gamma)$ as in the preceding. Writing down explicitly the condition for isomorphism between $\mathfrak{g}(\gamma)$ and $\mathfrak{g}(\rho)$ we obtain (in terms of descent)
\[ A^{-1}C(\gamma)A(\mu_1, \mu_2, \mu_3) = C(\rho) \quad A \in GL(\mathfrak{S}/\Sigma \mathfrak{P}e_i), \quad \mu_i \in \mathfrak{P}^*. \]

In particular we see that \([(123), 1]C(\gamma)[(132), 1] = C(\gamma^*) \] so \(\mathfrak{L}(\gamma) \cong \mathfrak{L}(\gamma^*)\). More generally we have

**Theorem.** Let \(P/\Phi\) be cyclic sixth degree with \(F/\Phi\) the cubic sub-extension. If \(\gamma, \rho\) are elements of \(F\) of norm 1, then \(\mathfrak{L}(\gamma) \cong \mathfrak{L}(\rho)\) if and only if \(\mathfrak{L}(\gamma)^* \cong \mathfrak{L}(\rho)^*\) (as algebras without involution).

**Proof.** One direction is clear. For the other, the condition \(\mathfrak{L}(\gamma)^* \cong \mathfrak{L}(\rho)^*\) is equivalent to a relation of the form \(\rho = \gamma\lambda\lambda^*\) for some \(i, 0 \leq i \leq 2\). The preceding discussion enables us to assume that \(i = 0\), i.e. that \(\rho = \gamma\lambda\lambda^*\). Observe that \(N_{P/\Phi}(\lambda) = 1\) (take \(N_{P/\Phi}\) of both sides) and set \(\epsilon = \gamma(\lambda^*\lambda^*)^{-1}\). Then \(\epsilon \epsilon^* = 1\) and we let \(E\) be the related triple described by Lemma 1 for \(\epsilon\), \(\epsilon^*, \epsilon^{*^2}\). A straightforward calculation shows that

\[ [1, E]^{-1}C(\gamma)[1, E]((\lambda\lambda^*)^{-1}, (\lambda^*\lambda^*)^{-1}, (\lambda\lambda^*)^{-1}) = C(\rho) \]

so \(\mathfrak{L}(\gamma) \cong \mathfrak{L}(\rho)\). q.e.d.

2. Special fields. As remarked above, our construction may be carried out over finite algebraic number fields. The results of [2] show that any \(D_{III}\) over such a field is split by a cyclic sixth degree extension \(P/\Phi\) and by a slight modification of the proof of Proposition 3 of [2] we may assume that \(P\) has no real primes. Let \(\mathfrak{L}\) be a non-Jordan \(D_{III}\) over \(\Phi\). Let \(\mathfrak{L}_\Phi\) (\(F\) as before) be the canonical \(D_{II}\) extension of \(\mathfrak{L}\). In the indicated reference it is also shown that \(\mathfrak{L}_\Phi\) is fixed under conjugation by a semilinear transformation \([1, (C_\gamma)]\) where \([1, (C_\gamma)]^2 = (\gamma^3, (\gamma^3)^3, (\gamma^3)^3), \gamma \in F, N_{P/\Phi}(\gamma) = 1\). Since \(\mathfrak{L}\) is a non-Jordan \(D_{III}\), \(\gamma \in N_{P/\Phi}(P^*)\).

Let \(\Delta = (P, t, \gamma^*)\), \(t = s^4\). Then \(\mathfrak{L}_\Phi\) has a realization as \(\mathfrak{K}(\Delta, J)\) which we can describe explicitly as follows:

Write \(\Delta = P + CP, C^2 = \gamma^3, \alpha C = C\alpha^4\), define a \(\Delta\)-module structure on \(\mathfrak{C}\) by setting \(x \cdot (\alpha + C\beta) = x\alpha + (xC\beta)\), and let \(-\) denote the involution \(\alpha + C\beta \rightarrow \alpha + C\beta^t\) in \(\Delta\). It follows from [3] that \(\mathfrak{L}_\Phi\) is isomorphic to the Lie algebra of all \(\Delta\)-linear transformations in \(\mathfrak{C}\) which are skew with respect to the nondegenerate-hermitian form

\[ f(x, y) = n(x, y) + Cn(x, yC^{-1})^t \]

on \(\mathfrak{C}/\Delta\). In case \(\mathfrak{L} = \mathfrak{L}(\gamma)\), then \(\{u_1, u_2, u_3, u_4\}\) is an orthogonal basis for \(\mathfrak{C}/\Delta\) and we compute

\[ f(u_i, u_i) = C_{\gamma^3}^i \quad 1 \leq i \leq 3 \]

\[ f(u_4, u_4) = C_{\gamma^3}^4. \]
In a forthcoming paper, the first author has shown that $f$ cannot have maximal Witt index. However, using the Hasse principle for hermitian forms of type $D$ we conclude that $f$ has Witt index 0 if and only if there is a real prime $p$ on $F$ with $\mathfrak{p}_p$ the compact real $D_4$.

**BIBLIOGRAPHY**