INTERIORITY OF A HOLOMORPHIC MAPPING ON THE SET OF ITS EXCEPTIONAL POINTS

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I. Introduction. A mapping $f: A \rightarrow B$ is said to be interior (or open) if for every open subset $U \subseteq A$, $f(U)$ is an open subset of $B$; it is said to be interior at a point $a \in A$ (or locally interior at $a \in A$) if for every open subset $U \subseteq A$ containing $a$, $f(a)$ is an interior point of $f(U)$. Clearly a mapping is interior if and only if it is locally interior everywhere on its domain of definition.

The result contained in this note is about the local interiority property of a holomorphic mapping on the set of its exceptional points. We shall restrict our attention to holomorphic mappings $f = (f_1(x), \ldots, f_n(x)): D \rightarrow \mathbb{C}^n$ where $D$ is a domain (open connected set) in $\mathbb{C}^n$. $\mathbb{C}^n = \mathbb{C}^1 \times \cdots \times \mathbb{C}^1$ where $\mathbb{C}^1$ is the extended plane of each one of the complex variables $x_i$. $f$ is said to be holomorphic when each one of the functions $f_i$ is holomorphic on $D$. Let $J(x)$ be the value of the Jacobian of $f$ at $x \in D$.

The set $E$ of exceptional points of $f$ is by definition $E = \{a \in D | a$ is not an isolated point of $f^{-1}(f(a))\}$.

II. Result. We recall that if $a \notin E$, $f$ is interior at $a$. In fact, if $a \notin E$ and $J(a) \neq 0$, the property follows immediately from the inverse function theorem ($f$ is a local homeomorphism); if $a \notin E$ and $J(a) = 0$, it follows from a theorem of Osgood [1] ($f$ maps finitely-to-one sufficiently small neighborhoods of $a$ onto neighborhoods of $b = f(a)$).

Our result pertains to the case $a \in E$:

THEOREM. Let $f: D \rightarrow \mathbb{C}^n$, $D \subseteq \mathbb{C}^n$, be a holomorphic mapping and let $E$ be the set of exceptional points of $f$, then the subset $E_0$ in $E$ such that $E_0 = \{x \in E | f$ is interior at $x\}$ is either the empty set or a set of isolated points.

PROOF. If $E$ is empty, $f$ is everywhere interior in $D$ as shown above. If $f$ is degenerate, i.e., $J(x) \equiv 0$, it is not difficult to show that $E = D$ and $E_0 = \{\emptyset\}$.

Let then $f$ be not degenerate and $E$ not empty. H. Cartan [2] proved that $E$ is an analytic set and $E \subseteq W = \{x \in D | J(x) = 0\}$. Com-

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plex-dimension \( (W) = n - 1 \) and complex-dimension \( (W' = f(W)) \leq n - 1 \). Let \( S = \{ S_1, \ldots, S_r \} \subset E \) be the set (finite) of irreducible local analytic varieties passing through a given arbitrary point \( a \in E \) and let \( V = \{ V_1, \ldots, V_s \} \) be the set (finite) of irreducible subvarieties in \( S \) which are associated with \( a \), meaning that \( f(V) = f(a) = b = (b_1, \ldots, b_n) \). Now we consider any one-complex-dimensional analytic plane \( II \) passing through \( b \) and not contained in \( W' \). Let

\[
\Pi = \{ y \in C^n | (y_1 - b_1)/\alpha_1 = \cdots = (y_n - b_n)/\alpha_n \}
\]

where \( \alpha_1, \ldots, \alpha_n \) are complex constants, be that plane. Obviously \( f^{-1}(\Pi) = \{ x \in D | (f_1(x) - b_1)/\alpha_1 = \cdots = (f_n(x) - b_n)/\alpha_n \} \).

This is an analytic set, consequently, [3], locally at the given point \( a \in E \) it consists of a finite set of irreducible analytic varieties which will be called \( \theta \). Clearly, \( \theta \supset V \) since \( f(V) = b \) and \( b \in II \).

**Case 1.** \( \theta = V \), then \( f \) is not locally interior at \( a \). Indeed, if \( N_a \subset D \) is a sufficiently small neighborhood of \( a \), \( b \) will be the only point in \( II \) contained in \( f(N_a) \); this proves that \( b = f(a) \) is on the boundary of \( (N_a) \).

**Case 2.** \( \theta \supset V \). This means that \( \theta = \{ V, \theta_1, \ldots, \theta_p \} \) where \( \theta_1, \ldots, \theta_p \) are the irreducible analytic varieties in the local decomposition of \( f^{-1}(\Pi) \) which are not contained in \( V \). Since \( \Pi \) is not contained in \( W' \), none of the \( \theta_i \) is contained in \( W \). Hence, each one of the \( \theta_i \) being mapped under \( f \) into \( II \) is itself of complex-dimension 1. This proves that the intersection of the \( \theta_i \) with \( E \) is a set \( E^* \subset E \) which consists of isolated points. In order for \( f \) to be locally interior at \( a \) it is necessary that for every \( II \) defined as above there exist varieties \( \theta \), Hence, the set \( E_0 \subset E \) of points where \( f \) is locally interior certainly satisfies \( E_0 \subset E^* \) (\( E^* \) was defined for a single \( II \)) and therefore \( E_0 \) contains at most isolated points. Q.E.D.

As an immediate corollary we obtain a result proved by R. Remmert [4].

**Corollary.** A holomorphic mapping \( f : D \to C^n, D \subset C^n \), is interior if and only if \( E \) is the empty set.

**III. Examples.** In order to show that the two possibilities for \( E_0 \) which were mentioned in the Theorem can actually occur, we give the two following examples.

**Example 1.** \( f = (y_1 = x_1x_2, y_2 = x_2) : C^2 \to C^2 \). Here \( J(x) = x_2, E = W = \{ x \in C^2 | x_2 = 0, x_1 \text{ arbitrary} \}, W' = f(E) = \{ 0' = (y_1 = y_2 = 0) \} \). It is clear that the set \( II = \{ 0' \}, \) where \( II = \{ y \in C^n | y_2 = 0, y_1 \text{ arbitrary} \} \), is not in the range of \( f \). Thus, \( \forall a \in E \) and any open set \( N_a \subset C^2 \exists a \in N_a \),
0' = f(a) is on the boundary of \( f(N_a) \). This proves that \( E_0 = \{ \emptyset \} \).

**Example 2.** \( f = (y_1 = x_1(x_3 - x_1), y_2 = x_1(x_2 + x_3), y_3 = x_1x_3x_3) : \mathbb{C}^3 \to \mathbb{C}^3 \). Here \( E = \{ x \in \mathbb{C}^3 \mid x_1 = 0, x_2 \text{ and } x_3 \text{ arbitrary} \} \). We shall show that \( f \) is locally interior at \( 0 \in E, 0 = (x_1 = x_2 = x_3 = 0) \), by proving that for arbitrarily small \( \varepsilon_1 > 0 \), \( \exists \delta_1 > 0 \), \( \forall y \in \text{boundary}(\Sigma') \) and \( 0 < \delta < \delta_1 \), \( \exists x \in \Sigma \exists f(x) = y \), where \( \Sigma \) and \( \Sigma' \) are open hyperspheres, respectively, centered at \( 0 \) and \( 0' \) with radius \( \varepsilon_1 \) and \( \delta \).

From the equations defining \( f \) we can derive:

1. \( x_1^4 - x_1^2(y_2 - 2y_1) + x_1y_2 + y_1(y_1 - y_2) = 0 \)
2. \( x_2 = (y_2 - y_1)/x_1 - x_1 \)
3. \( x_3 = y_1/x_1 + x_1 \)

Let us consider a surface \( \sigma = \{ y \mid |y_1|^2 + |y_2|^2 + |y_3|^2 = \varepsilon^2 \} \) where \( 0 < \varepsilon \ll 1 \). Our first step is to find a common upper bound for the roots \( x_i^* \), \( i = 1, \ldots, 4 \), of equation (1) when \( y \in \sigma \). We can write (1) as

\[(1') \quad x_1^2 = (y_2 - 2y_1)/2 \pm (y_2^2/4 - x_1y_2)^{1/2} \]

\( \forall y \in \sigma \), we obtain from (1')

\[
|x_1|^2 < \frac{3\varepsilon^2}{2} + \left( \frac{\varepsilon^2}{4} + |x_1| \varepsilon \right)^{1/2} < \frac{3\varepsilon^2}{2} + \left( \frac{\varepsilon^2}{4} \right)^{1/2} + (|x_1| \varepsilon)^{1/2} = 2\varepsilon^2 + (|x_1| \varepsilon)^{1/2}.
\]

Since \( \varepsilon \ll 1 \), it is not difficult to see that this inequality holds for

4. \( |x_1| < \varepsilon + o(\varepsilon^2) \) where \( o(\varepsilon^2) \) is of the order of \( \varepsilon^2 \) when \( \varepsilon \to 0 \).

Now let \( x_1^m \) be one of the four roots \( x_i^* \) whose absolute value is larger or equal to the absolute value of all the others. We want to find a lower bound for \( x_1^m \). To that purpose we introduced the following symmetric functions of the \( x_i^* \), obtained from (1):

\[ s_4 = x_1x_1x_1x_1 = y_1(y_1 - y_2), \]
\[ s_3 = x_1x_1x_1 + \cdots + x_1x_1x_1 = (\text{total of 4 terms}) = - y_3, \]
\[ s_2 = x_1^2 + \cdots + x_1^2 = (\text{total of 6 terms}) = y_2 - 2y_1. \]

Clearly:

\[
|x_1^m| \geq |s_4|^{1/4} \geq \left| y_1 \right|^{1/4} \left| y_1 \right| - \left| y_2 \right| \geq \left| y_2 \right|^{1/4},
\]
\[
|x_1^m| \geq |s_4/4|^{1/3} = \left| y_2/4 \right|^{1/3},
\]
\[
|x_1^m| \geq |s_4/6|^{1/2} \geq \left( \left| y_2 \right| - 2 \left| y_1 \right| \right)/6^{1/2}.
\]
Therefore
\[ |x_1^m| \geq \frac{|y_1|^{1/4} \left| y_1 - \frac{y_2}{4} \right|^{1/2} + \frac{1}{6} |y_1 - 2| y_1|^{1/2}}{3} \]

\( \forall y \in \sigma, \) it follows from this last inequality that \( |x_1^m| > \varepsilon^{3/2}/9. \) Hence, recalling (4), we have

(5) \[ \varepsilon^{3/2}/9 < |x_1^m| < \varepsilon + o(\varepsilon^2). \]

Finally from (2), (3) and using (5) we obtain:

\[ |x_2^m| \leq \frac{|y_2| + |y_1| + |x_1^m|^2}{|x_1^m|} < \frac{2\varepsilon^3 + \varepsilon^2 + o(\varepsilon^3)}{\varepsilon^{3/2}/9} = 9\varepsilon^{1/2} + o(\varepsilon^{3/2}), \]

\[ |x_3^m| \leq \frac{|y_1| + |x_1^m|^2}{|x_1^m|} < \frac{\varepsilon^3 + \varepsilon^2 + o(\varepsilon^3)}{\varepsilon^{3/2}/9} = 9\varepsilon^{1/2} + o(\varepsilon^{3/2}). \]

If \( \varepsilon \) is taken to be sufficiently small, then certainly

\[ |x_1^m| < \varepsilon + o(\varepsilon^2) < 10\varepsilon^{1/2}, \quad |x_2^m| < 9\varepsilon^{1/2} + o(\varepsilon^{3/2}) < 10\varepsilon^{1/2}, \]

\[ |x_3^m| < 9\varepsilon^{1/2} + o(\varepsilon^{3/2}) < 10\varepsilon^{1/2}. \]

In order to complete the required proof it is enough to put

\[ \varepsilon_1 = 10\varepsilon^{1/2} \quad \text{and} \quad \delta_1 = \varepsilon = 10^{-4} \times \varepsilon_1. \]

By using arguments similar to those given in the proof of the Theorem, it is possible to show that \( \forall a \in E \) and \( a \neq 0, f \) is not interior at \( a. \) Thus \( E_0 = \{0\} \neq \{0\}. \)

**References**


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