

A CHARACTERIZATION OF SPACES WITH VANISHING GENERALIZED HIGHER WHITEHEAD PRODUCTS

BY F. D. WILLIAMS¹

Communicated by R. H. Bing, November 27, 1967

A subject of recent investigation in homotopy theory has been the study of generalized higher order Whitehead products, cf. [1] and [3]. Let us say that a space, X , has property P_n if for any $f_1, \dots, f_n, f_i: SA_i \rightarrow X$, we have $0 \in [f_1, \dots, f_n]$, where $[f_1, \dots, f_n]$ denotes the set of all n th order Whitehead products of f_1, \dots, f_n , as defined in [3]. Thus $0 \in [f_1, \dots, f_n]$ means that $f_1 \vee \dots \vee f_n: SA_1 \vee \dots \vee SA_n \rightarrow X$ can be extended to some $F: SA_1 \times \dots \times SA_n \rightarrow X$. (We note at this point that it is an unresolved conjecture as to whether X has property P_n implies that 0 is the *only* element of $[f_1, \dots, f_n]$.) Now if X is an H -space, then X possesses property P_n for all n , [3]. Thus multiplicative properties of X itself are too strong to distinguish among the various properties P_n . On the other hand, it follows from results of [1] and [4] that a space has property P_2 if and only if its loop space is homotopy-commutative. In Theorem 1 below, we shall extend this result to characterize those spaces which have property P_n in terms of higher homotopy-commutativity properties of their loop spaces. Since we shall wish to be able on occasion to replace a loop space by an equivalent CW-monoid, we shall restrict our attention to the category of countable CW-complexes.

The higher homotopy-commutativity properties we need are described in the following definition which was introduced in [7].

DEFINITION. An associative H -space, Y , is called a C_n -space provided that there exist maps $Q_i: C(i-1) \times Y^i \rightarrow Y, 1 \leq i \leq n$, such that:

- (1) $Q_1: C(0) \times Y \rightarrow Y$ is the identity;
- (2) $Q_i([\mu, \nu] \circ d_p(r, s), y_1, \dots, y_i) = Q_p(r, y_{\mu(1)}, \dots, y_{\mu(p)}) \cdot Q_q(s, y_{\nu(1)}, \dots, y_{\nu(q)})$, for (p, q) -shuffles (μ, ν) , where $p+q=i, r \in C(p-1)$, and $s \in C(i-p)$; and
- (3) if e denotes the identity of Y , and if $y_i = e$, then

$$Q_i(T, y_1, \dots, y_i) = Q_{i-1}(D_j(T), y_1, \dots, y_j, \dots, y_i).$$

Here $C(i)$ is the convex linear cell described in [2], namely the convex hull of the orbit of the point $(1, \dots, n+1)$ under permutation of the coordinates in R^{n+1} . The map $d_p: C(p-1) \times C(i-p) \rightarrow C(i)$ is given by $d_p(x_1, \dots, x_p, y_1, \dots, y_{i-p+1}) = (x_1, \dots, x_p, y_1 + p, \dots, y_{i-p+1} + p)$,

¹ Supported by the National Science Foundation Grant GP-6318.

the map $D_j: C(i) \rightarrow C(i-1)$ is as defined in [2], and $[\mu, \nu]: C(i) \rightarrow C(i)$ is induced by the actions of the shuffle (μ, ν) on R^{n+1} by permutation of coordinates.

Note that a C_2 -space is just a homotopy-commutative monoid. Furthermore, the usual proof that the loop space of an H -space is homotopy-commutative extends to yield the fact that such a loop space is a C_n -space for every n . (It is known that the converse of this fact is false.) We recall from [7] the main theorem on C_n -spaces, which will be used in the proof of Theorem 1.

THEOREM 0. *An associative H -space, Y , is a C_n -space if and only if the Hopf fibration for Y , $p_1: Y * Y \rightarrow SY$, extends to a fibration $p_n: E_n \rightarrow (SY)_n$, where $(SY)_n$ denotes the n -fold reduced product space of the suspension of Y .*

The idea of C_n -commutativity is somewhat analogous to Stasheff's theory of A_n -associativity, [5]. Thus the reduced product spaces $(SY)_n$ stand in relation to commutativity much as the projective spaces $XP(n)$ relate to associativity. The proof of the "only if" part of Theorem 0 is inspired by Stasheff's work, and is accomplished by a direct construction in the Dold-Lashof vein. The reverse implication uses the connecting map $r: \Omega(SY)_n \rightarrow Y$ together with the fact that Y is C_n in $\Omega(SY)_n$. Here Y is regarded as a subspace of $\Omega(SY)_n$ via the composition $Y \xrightarrow{j} \Omega SY \subset \Omega(SY)_n$, where j is the usual inclusion. The notion of a subspace being C_n in a containing space is a natural extension of the well-known concept of a subspace being homotopy-commutative in a larger space.

The definition of C_n -space permits us to state the main theorem of this note.

THEOREM 1. *A space possesses property P_n if and only if its loop space is a C_n -space.*

The proof of this theorem, which will be outlined below, is based on the following theorem.

THEOREM 2. *A monoid, Y , is a C_n -space if and only if the inclusion $i: SY \rightarrow B_Y$ extends to a map $a: (SY)_n \rightarrow B_Y$.*

The main theorem follows readily from Theorem 2 as follows. Let X be a space and Y a monoid for which there exists a strongly homotopy-multiplicative homotopy equivalence $f: Y \rightarrow \Omega X$, as in [6]. Then f induces a homotopy equivalence $g: B_Y \rightarrow X$. Let c be the evaluation map $c: S\Omega X \rightarrow X$. Then (Sf, g) is a homotopy equivalence of the map i with the map c . Now any map $h: SA \rightarrow X$ factors as $c \cdot S\tilde{h}$,

where \tilde{h} is the adjoint of h , and hence h factors up to homotopy through i . Now if ΩX is a C_n -space, then so is Y , and hence $0 \in [i, \dots, i]$ (n factors), by Theorem 2. Consequently if $f_i: SA_i \rightarrow X$, $1 \leq i \leq n$, then $0 \in [f_1, \dots, f_n]$, and thus X satisfies property P_n . Conversely, it follows from Theorem 2.8 of [3] that $0 \in [i, \dots, i]$ implies that i extends to all of $(SY)_n \rightarrow B_Y$.

The proof of Theorem 2 goes as follows. The "if" part is easily obtained from Theorem 0 by taking $p_n: E_n \rightarrow (SX)_n$ to be the fibration induced by $a: (SY)_n \rightarrow B_Y$ from $\pi: \mathcal{E}_\infty \rightarrow B_Y$, the Dold-Lashof universal fibration for Y . The converse is the nontrivial implication and is accomplished by using the maps Q_i to map E_n to \mathcal{E}_{n+1} , the total space of the $(n+1)$ th stage of the Dold-Lashof construction, in fiber-wise fashion, thus inducing a map in the base spaces $(SY)_n \rightarrow YP(n) \subset B_Y$. The details are rather lengthy and will appear elsewhere.

REFERENCES

1. M. Arkowitz, *The generalized Whitehead product*, Pacific J. Math. 12(1962), 7-23.
2. R. J. Milgram, *Iterated loop spaces*, Ann. of Math. 84(1966), 386-403.
3. G. J. Porter, *Higher order Whitehead products*, Topology 3(1965), 123-135.
4. J. Stasheff, *On homotopy Abelian H-spaces*, Proc. Cambridge Philos. Soc. 57(1961), 734-745.
5. ———, *Homotopy associativity of H-spaces*, Trans. Amer. Math. Soc. 108(1963), 275-312.
6. M. Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A 33(1960), 257-269.
7. F. D. Williams, Ph.D. thesis, University of Wisconsin, Madison, Wis., 1965.

NEW MEXICO STATE UNIVERSITY