

MODIFICATION SETS OF DENSITY ZERO¹

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Let R, Z, T denote the real line, the integers, and the unit circle, respectively. A set $E \subset R$ will be called a *modification set in R* if to every $f \in L^1(R)$ there corresponds a singular bounded Borel measure μ on R whose Fourier transform $\hat{\mu}$ coincides with \hat{f} in the complement of E . In other words, the Fourier transform of every absolutely continuous measure can be modified on E alone so that the resulting function is the Fourier transform of a singular measure. Modification sets E in Z are defined similarly: to every $f \in L^1(T)$ there should correspond a bounded singular measure μ on T whose Fourier coefficients satisfy $\hat{\mu}(n) = \hat{f}(n)$ for every integer n which is not in E .

The existence of "small" modification sets in locally compact abelian groups has been established in [1]. However, when applied to Z or R , the theorem of [1] can only yield modification sets of positive (though arbitrarily small) lower density. In the present note this result is improved to yield sets of density zero.

A set $E \subset R$ is said to have density zero if $(2t)^{-1}m(E \cap [-t, t]) \rightarrow 0$ as $t \rightarrow \infty$, where m denotes Lebesgue measure. If $E \subset Z$, the requirement is that the number of elements of E in $[-N, N]$, divided by $2N$, should tend to 0 as $N \rightarrow \infty$.

THEOREM 1. *There are modification sets of density zero in R .*

THEOREM 2. *If E is a modification set in R then $E \cap Z$ is a modification set in Z .*

THEOREM 3. *There are modification sets of density zero in Z .*

REMARK. Modification sets can of course not be *too* small. For instance, every modification set in R has infinite measure (Plancherel); no lacunary set in Z is a modification set; no set of positive integers is a modification set (F. and M. Riesz). On the other hand, largeness is not enough: Theorem 2 shows that the complement of Z in R is not a modification set.

PROOF OF THEOREM 1. Choose integers $\lambda_1, \lambda_2, \lambda_3, \dots$ so that $\lambda_1 = 10, \lambda_k \geq 4\lambda_{k-1}$. Let A_k be the set of all numbers of the form

$$(1) \quad \pm \lambda_k + \epsilon_{k-1} \lambda_{k-1} + \dots + \epsilon_1 \lambda_1$$

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where $\epsilon_i = 1$ or 0 or -1 , let B_k be the union of all intervals of length $2k$ whose centers are in A_k , and put $E = B_1 \cup B_2 \cup B_3 \cup \dots$.

[Given $t > 10$, let $k = k(t)$ be the largest integer such that $\lambda_k \leq 2t$. Then $E \cap [-t, t] \subset B_1 \cup \dots \cup B_k$. Since A_i has $2 \cdot 3^{i-1}$ points, $m(B_i) \leq 4i \cdot 3^{i-1}$. Hence

$$\frac{m(E \cap [-t, t])}{2t} \leq \frac{1}{2t} \sum_{i=1}^k m(B_i) < \frac{2k \cdot 3^k}{\lambda_k} \leq \frac{4k}{5} \cdot \left(\frac{3}{4}\right)^k$$

which tends to 0 as t (and hence k) tends to ∞ . Thus E has density zero.

For $k = 1, 2, 3, \dots$, let σ_k be the measure on T whose Fourier series is the formal expansion of the Riesz product

$$(2) \quad d\sigma_k(x) \sim \prod_{j=k}^{\infty} (1 + \cos \lambda_j x).$$

Then σ_k is a bounded, positive, continuous, and singular measure on T [2, p. 209] and $\hat{\sigma}_k(n) = 0$ unless $n \in \{0\} \cup A_k \cup A_{k+1} \cup \dots$.

Now choose $f \in L^1(R)$ so that \hat{f} has compact support and two continuous derivatives. Then $x^2 f(x)$ is bounded, so that $\sum |f(x - 2\pi j)|$, $j \in Z$, is a continuous periodic function. Fix k so that $\hat{f}(t) = 0$ whenever $|t| > k$. If we average the left side of (3) below over $-\pi \leq s \leq \pi$, and apply Fubini's theorem, we see that there exists an s (fixed from now on) such that

$$(3) \quad \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} |f(x - s - 2\pi j)| d\sigma_k(x) \leq \int_{-\infty}^{\infty} |f(y)| dy = \|f\|_1.$$

Define a measure μ on R by requiring that

$$(4) \quad \int_{-\infty}^{\infty} g d\mu = \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} g(x - s - 2\pi j) f(x - s - 2\pi j) d\sigma_k(x)$$

for every bounded continuous g . Then μ is a singular measure on R whose total variation satisfies $\|\mu\| \leq \|f\|_1$, by (3). The Poisson summation formula now gives

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-itx} d\mu(x) &= \int_{-\pi}^{\pi} \sum_{j=-\infty}^{\infty} e^{-it(x-s-2\pi j)} f(x - s - 2\pi j) d\sigma_k(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \hat{f}(t - n) e^{-in(x-s)} d\sigma_k(x) = \sum_{n=-\infty}^{\infty} \hat{\sigma}(n) \hat{f}(t - n) e^{in s} \end{aligned}$$

which is the same as

$$(5) \quad \hat{\mu}(t) = \hat{f}(t) + \sum_{n \neq 0} \delta(n) \hat{f}(t-n) e^{in\pi} \quad (t \in R).$$

In the last sum, $\delta(n) = 0$ unless $n \in A_k \cup A_{k+1} \cup \dots$, and $\hat{f}(t-n) = 0$ if $|t-n| \geq k$. Hence $\hat{\mu}(t) = \hat{f}(t)$ except possibly in $B_k \cup B_{k+1} \cup B_{k+2} \cup \dots$ which is a subset of E .

To conclude the proof, let f be an arbitrary member of $L^1(R)$. Then $f = \sum f_n$ where $\sum \|f_n\|_1 < \infty$ and each \hat{f}_n has compact support and two continuous derivatives. The preceding step shows that there are singular measures μ_n with $\|\mu_n\| \leq \|f_n\|_1$, such that $\hat{\mu}_n(t) = \hat{f}_n(t)$ outside E . The series $\sum \mu_n$ then converges in the total variation norm to a measure μ which is therefore also singular, and if t is not in E we have

$$(6) \quad \hat{\mu}(t) = \sum \hat{\mu}_n(t) = \sum \hat{f}_n(t) = \hat{f}(t).$$

Thus E is a modification set in R .

PROOF OF THEOREM 2. Let E be a modification set in R . Choose $f \in L^1(T)$, regard f as a member of $L^1(R)$ which vanishes outside $[-\pi, \pi)$, and let μ be a singular measure on R such that $\hat{\mu}(t) = \hat{f}(t)$ outside E . For $V \subset [-\pi, \pi)$, define $\sigma(V) = \sum \mu(V - 2\pi j)$, $j \in Z$. Then σ is a singular measure on T , and $\delta(n) = \hat{\mu}(n)$ for every $n \in Z$. If $n \in Z$ and $\hat{f}(n) \neq \delta(n)$ it follows that $n \in E \cap Z$. So $E \cap Z$ is a modification set in Z .

PROOF OF THEOREM 3. If E is one of the sets constructed in the proof of Theorem 1 then $E \cap Z$ has density zero in Z . Hence Theorem 3 follows from Theorem 2.

REFERENCES

1. Walter Rudin, *Modifications of Fourier transforms*, Proc. Amer. Math. Soc. (to appear).
2. Antoni Zygmund, *Trigonometric series*, 2nd ed., Vol. I, Cambridge Univ. Press, New York, 1959.

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