ALMOST EVERYWHERE CONVERGENCE OF POISSON INTEGRALS ON GENERALIZED HALF-PLANES

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1. Introduction. A classical theorem of Fatou states that if \( f \) is an \( L^p \) function on the line (circle), \( p \geq 1 \), and if the harmonic function \( F \) on the upper half-plane (disk) is the Poisson integral of \( f \), then \( F(z) \to f(x) \) as \( z \to x \) nontangentially for a.e. \( x \) on the line (circle).

Generalizations in several directions have recently been found, e.g. [1], [2], [4], [6]. Our result, stated precisely below, is Fatou's theorem for generalized upper half-planes holomorphically equivalent to bounded symmetric domains and functions of type \( L^p, p > 1 \), or locally of type \( L \log + L \). Details will appear elsewhere.

In \( \S 2 \), we sketch the setting and state our result explicitly. The proof is case-by-case, and includes the case of the exceptional domains; \( \S 3 \) is devoted to a sketch of the proof in a typical case.

2. The theorem. Let \( D \) be a generalized upper half-plane, i.e. 

\[
D = \{(z, w) \in V_1 \times V_2 : \text{Im } z - \Phi(w, w) \in \Omega\},
\]

where \( V_1 \) is a complex vector space with a given real form, \( V_2 \) is a complex vector space, \( \Omega \subset \text{Re } V_1 \) is an open cone, and \( \Phi : V_2 \times V_2 \to V_1 \) is hermitian symmetric bilinear with respect to \( \text{Re } V_1 \) such that \( \Phi(w, w) \in \Omega \). When \( \Omega \) is a domain of positivity and \( \Phi \) satisfies certain symmetry and homogeneity properties, \( D \) is holomorphically equivalent to a bounded symmetric domain [5]. The distinguished boundary of \( D \) is

\[
B = \{(z, w) : \text{Im } z - \Phi(w, w) = 0\}.
\]

We identify \( B \) with \( \text{Re } V_1 \times V_2 \) by associating to \((x + i\Phi(w, w), w)\) the pair \((x, w)\). There is a nilpotent group \( \mathcal{N} \) of automorphisms of \( D \) which acts transitively on \( B \) and is also equal to \( \text{Re } V_1 \times V_2 \) as a set. Haar measure on \( \mathcal{N} \) is the induced Euclidean measure.

The Poisson kernel, \( P(u, \xi) \), is defined on \( B \times D \), and the Poisson integral of a function \( f \) on \( B \) is

\[
F(\xi) = \int_B f(u) P(u, \xi) du.
\]
For details of the above, see [3]. For \( u \in B, t \in \Omega \) we write \( u_t = u + (it, 0) \). Also, let \( I \) be the base point in \( \Omega \).

**Theorem 1.** Let \( D \) be a generalized upper half-plane holomorphically equivalent to a bounded symmetric domain. Suppose that \( f \in L^p(B), p > 1, \) or that \( f \in L \log^+ L \) locally and is bounded off a bounded set. Then

\[
\lim_{r \to 0} F(u_t) \to f(u) \quad \text{for almost every } u \in B.
\]

There are more general types of convergence. We say that \( t \in \Omega \) approaches \( 0 \) restrictedly if \( t \) is constrained to lie in a proper subcone of \( \Omega \). And we say that \( u_t \to u_0 = g_0 \cdot 0 \) admissibly if \( u_t \) stays in some

\[
\Gamma_a(u_0) = \{ g \in G : (it, 0) : g = (a, c), \max\{ |a|, |c|^2 \} < a |t| \}.
\]

**Theorem 2.** Under the hypothesis of Theorem 1, \( F(u_t) \to f(u_0) \) for a.e. \( u_0 \in B \) as \( u_t \to u_0 \) admissibly and restrictedly.

3. The proof. The proof for domains \( D \) which are tube domains, i.e. for which \( V_2 = 0 \), is contained in [6]. The remaining domains, with the exception of one of dimension 16, fall into two large classes, type I and type IIIb. We indicate the proof of Theorems 1 and 2 for domains of type I. The complete proof requires only slight modification.

There is a domain of type I for each pair of integers \( n, m, n > 0, m \geq 0 \). As a bounded domain, it is realized as the space of complex \( n \times (n + m) \) matrices \( \xi \) satisfying \( \xi^* \xi < I \). In the realization we consider, \( V_1 \) is the complexification of the real vector space of hermitian symmetric \( n \times n \) matrices, \( V_2 \) is the space of complex \( n \times m \) matrices, \( \Omega \) is the cone of positive definite matrices, and \( \Phi(w, w_1) = ww_1^* \). Thus

\[
D = D_{n,m} = \{(x + iy, w) : y - ww^* > 0 \}.
\]

The Poisson integral \( F((g \cdot 0)_t) \) is shown to be dominated by a sum of maximal functions \( f^*_t((g \cdot h)_t)(g), g \in \mathcal{G} \). We define these.

Let \( (j) \) and \( (k) \) be, respectively, \( n \)-tuples and \( m \)-tuples of nonnegative integers and \( t > 0 \).

\[
R^t_{(j)} = \{ r = (r_1, \ldots, r_n) \in E_n : |r_i| \leq 2^{\delta_{ij}}t \},
\]

\[
S^t_{(k)} = \{ s \in E_m : |s_i| \leq 2^{\delta_{ik}}t \}.
\]

Every \( x \in \text{Re } V_1 \) can be written in the form \( x = k^{-1}d(r)k, k \in U(n), r \in E_n \), where \( d(r) \) is the diagonal \( n \times n \) matrix whose entries are the \( r_i \).

And every \( w \in V_2 \) can be written \( w = u d(s)v, u \in U(n), v \in U(m) \).
$s \in E_m$, where $\tilde{d}(s)$ is the $n \times m$ diagonal-form matrix whose entries are the $s_i$. We set

$$H_{(j),(k)} = \{(x, w) = (k^{-1}d(r)k, u\tilde{d}(s)v) : r \in \mathbb{R}_{(j)}, s \in S_{(k)}^{\frac{1}{2}}\}.$$  

(If $m \geq n$, the rectangles $S$ lie in $E_n$, but this makes no difference.) There is defined for each $(j)$ and $(k)$ a sequence of neighborhoods

$$U(n) = N_{(j),(k)} \supset N_{(j),(k)} \supset \cdots \supset \{I\},$$

and we define

$$E_{(j),(k)} = \{(x, w) = (k^{-1}d(r)k, u\tilde{d}(s)v) \in H_{(j),(k)} : ku \in N_{(j),(k)}\},$$

$$f_{(j),(k)}(g) = \sup_{t>0} |E_{(j),(k)}|^{-1} \int_{E_{(j),(k)}} f(g) \, dh.$$  

We abuse notation now by writing $t$ for both a positive number and the matrix $tI$.

**Lemma 1.**

$$\sup_{t>0} F((g \cdot 0), t) \leq A \sum_{(j),(k) \in E_{(j),(k)}^{1-I}} 2^{-[(1/2)](f+i[k])} \sum_{i=1}^{L} f_{(j),(k)}(g),$$

where $L$ depends only on $m$ and $n$.

**Proof.** It is enough to prove the inequality when $g = 0$. We notice that

(1) $$P((x, w), (it, 0)) = P_t(x, w) = \left\{ \frac{\det t}{|\det(x + it[wv^* + I])|^2} \right\}^{n+m}.$$  

The method of proof is to compare the size of $P_t(x, u) \in E_{(j),(k)}^{1-I} - E_{(j-1),(k-1)}$, with the size of $|E_{(j),(k)}|$. (The $t^{1/2}$ factor in the definition of $E_{(j),(k)}$ is due to the $wv^*$ term in (1).) The necessity for considering the neighborhoods $N_0 \supset N_1 \supset \cdots$ may be seen by considering a special case.

In particular, if $x = k^{-1}d(2^j, 0, \cdots, 0)k$, $w = u\tilde{d}(2^{j/2}, 0, \cdots, 0)v$, then $|\det(x + it[wv^* + I])|^2$ depends on $ku$, ranging from $2^{2j} + (2^j + 1)^2$ when $ku = I$ to $(2^{j+1})(2^j + 1)^2$. The proof that the number of $N_j$ that need be considered is finite involves an induction, and is complicated.

The proof of Theorem 1 is now routine once one establishes

**Lemma 2.** $f_{(j),(k)} \in L_p \iff A_p \|f\|_p$, where $A_p$ is independent of $(j)$, $(k)$ and $l$.  

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Proof. It is not hard to show that

\[ f_{(j)(k)}(g) \leq A \frac{\mathcal{J}(j)(k)(g; k, u, v)x(ku)vdudk}{\int \int \int x(ku)vdudk}, \]

where \( x \) is the characteristic function of \( N_{(j)(k)} \) and

\[ f_{(j)(k)}(g; k, u, v) = \sup_{t > 0} \left| R_{(j)} |^{-1} S_{(k)}^{1/2} \int_{B_{(j)}} \int_{S_{(k)}} f(g \cdot (k^{-1}d(r)k, ud(s)v))dsdr. \]

The function \( f(\cdot ; k, u, v) \) may be thought of as giving maximal averages over \( m+n \)-dimensional rectangles "pointed" in the direction determined by \( u, v \) and \( k \). Now the subgroup

\[ \mathcal{H}_{k, u, v} = \{ h = (k^{-1}d(r)k, ud(s)v): r \in E_n, s \in E_m \} \]

of \( \mathfrak{M} \) is isomorphic to \( E_n \times E_m \), and so \( \mathcal{J}(j)(k) \) restricted to the coset \( g \cdot \mathcal{H}_{k, u, v} \) is an ordinary maximal function. Thus

\[ \int_{\mathcal{H}_{k, u, v}} | f_{(j)(k)}(gh; k, u, v) |^p dh \leq B_p \int_{\mathcal{H}_{k, u, v}} | f(gh) |^p dh. \]

Integrating over \( \mathfrak{M}/\mathcal{H} \) on both sides of (3), one has

\[ \int_{\mathfrak{M}} | f(gh; k, u, v) |^p dg \leq B_p \int_{\mathfrak{M}} | f(g) |^p dg. \]

This, together with (2), proves the lemma.

The proof that \( ||f_{(j)(k)}||_1 \leq A_{\delta} ||f||_L \) depends on the analogous result for ordinary maximal functions. To prove Theorem 1 in the case \( f \in L^1 \) would involve establishing a weak-type inequality,

\[ | \{ g: | f_{(j)(k)}(g) | > \delta \} | < A_{\delta}^{-1} ||f||_1. \]

Since the weak-type inequality for rectangular maximal functions cannot be "rotated" the way norm inequalities can, further analysis is necessary. This analysis has been performed by E. M. Stein (see [7]) and the author; and will appear.

We conclude by noting that Theorem 2 is a consequence of Theorem 1 and the following result, which is a slight extension of the corresponding result in the tube domain case.
Lemma 3. Suppose that $u_t \to u_0$ restrictedly and admissibly. Let $\lambda > 0$ be the smallest eigenvalue of $t$. Then, for any $u' \in B$,

$$P(u', u_t) \leq AP(u', (u_0)_t).$$

References