SOME RESULTS ON ONE-RELATOR GROUPS

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The results in this note arose from considering the question: What are the abelian subgroups of a one-relator group? The additive group of p-adic rationals and the free abelian group of rank 2 are certainly subgroups of a one-relator group. For example in

$$G = \mathrm{gp}\{a, b; a^{-1}b^{2p}ab^{-2}\}$$

the infinitely generated subgroup

$$H = \mathrm{sgp}\{b^2, a^{-1}b^2a, a^{-2}b^2a^2, \cdots\}$$

is isomorphic to the additive group of p-adic rationals, and

$$K = \mathrm{sgp}\{b^2, b^{-1}a^{-1}ba\}$$

is free abelian of rank 2. In 1964 Gilbert Baumslag [1] conjectured that the additive group of rationals is not a subgroup of a one-relator group. That this conjecture is correct follows from the following theorem.

THEOREM 1. *Let G be a torsion-free one-relator group. Then no nontrivial element of G has more than finitely many prime divisors. Moreover a nontrivial element is not divisible by more than finitely many powers of a prime p if p is greater than the length of the relator.*

REMARK. An element g of a group is divisible by an integer n, or has a divisor n, if g has an nth root in the group. By the length of the relator is meant the letter length of the relator as a word in a free group.

R. C. Lyndon [7] has shown that the cohomological dimension of a torsion-free one-relator group is $\leqq 2$. Now the cohomological dimension of a free abelian group of rank n is n, and of a direct product of an infinite cyclic group with a noncyclic locally cyclic group is >2, (see [6], [2]). Since the cohomological dimension of a subgroup is less than or equal to the cohomological dimension of the group, it follows that the only abelian subgroups of a torsion-free one-relator group are free abelian of rank $\leqq 2$ or locally cyclic subgroups in which every nontrivial element is divisible by at most finitely many primes.

The proof of Theorem 1 uses the usual argument of the Freiheitssatz (see [9]) together with the following ideas.

DEFINITION. Let H be a subgroup of G and p a prime. Then H is

p-pure in G if for each $g\in G$ and positive integer r, $g^{p^r}\in H$ implies that there exists an element $h\in H$ with $g^{p^r}=h^{p^r}$.

LEMMA 1. *Let $C=\{A * B;\ J\}$ be the generalized free product of two groups A, B amalgamating a subgroup J. If J is a p-pure subgroup of the factors A, B then A, B are p-pure subgroups of C. Moreover a nontrivial element of C is divisible by all powers of the prime p only if a nontrivial element of A or B is divisible by all powers of the prime p.*

The concept of a p-pure subgroup is the appropriate tool for proving Theorem 1, and indeed the whole argument turns on the following key lemma.

LEMMA 2. *Let G be a one-relator group. Then any subset of the generators of G generates a p-pure subgroup of G where p is any prime greater than the length of the relator.*

For one-relator groups with elements of finite order one can say much more.

THEOREM 2. *Let G be a one-relator group with torsion. Then the centralizer of every nontrivial element of G is cyclic.*

The proof of Theorem 2 is similar to the proof of Theorem 1 except instead of using p-pure subgroups one uses μ-subgroups defined as follows:

DEFINITION. Let H be a subgroup of G. Then H is a μ-subgroup of G if for all $g\in G$,

$$g^{-1}Hg \cap H \neq 1 \quad \text{implies} \quad g \in H.$$

LEMMA 3. *Let $C=\{A * B;\ J\}$ where J is a μ-subgroup of the factors A, B. Then A, B are μ-subgroups of C. If in A, B the centralizer of every nontrivial element is cyclic, then in C the centralizer of every nontrivial element is cyclic.*

LEMMA 4. *Let G be a one-relator group with torsion. Then any subset of the generators of G generates a μ-subgroup of G.*

The proof of Lemma 4 is not easy and depends on the following seemingly powerful result.

THEOREM 3. *Let $G=\mathrm{gp}\{a, b, \cdots;\ R^n\}$, $n>1$, where R is cyclically reduced. Suppose that two words $W=W(a, b, \cdots)$, $V=V(b, \cdots)$, where W is a freely reduced word containing a nontrivially and V does not contain a, define the same element of G. Then W contains a subword which is identical with a subword of $R^{\pm n}$ of length greater than $(n-1)/n$ times the length of R^n.*

This result tells us something about the actual spelling of words representing elements of G.

COROLLARY 1. *The word problem and the extended word problem are solvable in G.*

The proof of this result by W. Magnus (see [9]) is a rather complicated process and it does not show that in the special case of less than $\frac{1}{6}$ groups (see [3]) a much simpler solution is possible. Theorem 3, however, provides the simplest of algorithms.

COROLLARY 2. *Let F be a free group on a set X of generators and let $r \in F$ and $N = \{r^n\}^F$, $n > 1$. Let G and H be generated by subsets Y and Z of X. Then GHN is a recursive subset of F.*

This solves a problem of R. C. Lyndon (see [8, Problem 3.6]) in the case where N is the normal closure in F of a proper power. Lyndon points out that a solution of this problem enables one to considerably extend the Magnus solution of the word problem for one-relator groups.

The Freiheitssatz for groups with torsion is an immediate consequence of Theorem 3, and may be generalized in several directions, for example:

COROLLARY 3. *Let $H = \mathrm{sgp}\{a^\beta, b, \cdots\}$ be a subgroup of G where a, b are nontrivial in R. Then H is a free group freely generated by a^β, $b, \cdots$ if β is any integer $> 2\alpha$ where α is the largest absolute value of an a-exponent in R^n.*

This extends a result of N. S. Mendelsohn and Rimhak Ree [10]. If $n \geqq 8$ then the group is a less than $\frac{1}{8}$ group and from results of M. Greendlinger [4] the conjugacy problem is solvable in G if $n \geqq 8$. In this direction see also M. Greendlinger [5] for less than $\frac{1}{6}$ groups, and V. V. Soldatova [11] for a subclass of less than $\frac{1}{4}$ groups. By using a generalization of μ-subgroups one may prove

THEOREM 4. *The conjugacy problem is solvable in one-relator groups with torsion.*

Complete details, extensions, and applications of the results in this note will be submitted in a paper later.

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