FREDHOLM MAPS AND $\mathrm{GL}_c(E)$-STRUCTURES

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Many of the difficulties in the study of functions between infinite dimensional Banach spaces disappear when one considers only perturbations of a fixed, well behaved, map by a class of maps with some finiteness condition on their range, for example compact perturbations of the identity map as in the Leray-Schauder theory. The results stated below are intended to indicate how this procedure can be extended to study maps between Banach manifolds. In particular it can be used to describe the homotopy properties of Fredholm maps, introduced by Smale in [5].

These results are contained in the author's Oxford doctoral thesis, written under the supervision of M. F. Atiyah. It is a great pleasure to be able to thank Professor Atiyah, and also Professor J. Eells for all their help and encouragement.

A version of Theorem 2 was proved independently by A. J. Tromba who used it to develop an oriented degree theory for proper Fredholm maps which he applied to give a proof of the Schauder existence theorem for quasi-linear elliptic equations. A detailed discussion of all these results is intended in a future joint publication with A. J. Tromba.

Throughout, E and F will denote infinite dimensional Banach spaces, and X a paracompact space. A C^p-smooth manifold will mean a C^p Banach manifold which admits C^p partitions of unity. For background material and an exhaustive bibliography see the survey article by Eells [3].

1. Linear theory. The nonlinear theory is based on the linear theory sketched here.

$L(E, F)$ will denote the Banach space of bounded linear maps $T: E \to F$, $\Phi_n(E, F)$ the subspace of Φ_n-operators (i.e. Fredholm operators of index n), $\mathrm{GL}(E)$ the group of units in $L(E, E)$, $\mathrm{GL}_c(E)$ the subgroup of $\mathrm{GL}(E)$ consisting of elements of the form $I+\alpha$, where α is compact, and $\mathrm{GL}_F(E)$ the corresponding group with α of finite rank. A vector bundle map which is a Φ_n-operator on each fibre will be called a 'Φ_n bundle map'.

PROPOSITION 1. *Let $\pi: B \to X$ be an E-vector bundle.*

(i) *A Φ_0 bundle map $f: B \to X \times E$ over the identity map of X induces a unique $\mathrm{GL}_F(E)$-structure $\{\pi, f\}$ on π such that, in a trivialization of*

$\{\pi, f\}$, *f is represented in the form* $(x, v) \mapsto (x, v+\alpha(x)v)$, $x \in X$, $v \in E$, *where* $\alpha(x): E \to E$ *has finite rank.*

(ii) *If* π_0 *is a* $GL_F(E)$*-structure on* π, *there is a* Φ_0 *bundle map* $f: B \to X \times E$ such that $\{\pi, f\} = \pi_0$.

(iii) *There are corresponding statements to* (i) *and* (ii) *for* $GL_c(E)$*-structures on* π, *again using* Φ_0 *bundle maps.*

Let $VB(X, E)$, $K(X, E)$, $K_F(X, E)$ denote respectively the set of equivalence classes of E-vector bundles over X with groups $GL(E)$, $GL_c(E)$, $GL_F(E)$. Taking π as the trivial bundle in (iii) above, we obtain a map u: $[X, \Phi_0(E, E)] \to K(X, E)$.

COROLLARY. *The sequence*

$$[X, GL_c(E)] \to [X, GL(E)] \to [X, \Phi_0(E, E)] \xrightarrow{u} K(X, E) \to VB(X, E)$$

where the first two maps are induced by the inclusions, and the last is the forgetful map, is an exact sequence of sets with distinguished elements.

This is a version of the exact sequence of Atiyah and Jänich; see also Neubauer [4].

THEOREM 1. *The forgetful map* $K_F(X, E) \to K(X, E)$ *is bijective.*

Theorem 1 is a direct consequence of Proposition 1. It follows that the inclusion of $GL_F(E)$ into $GL_c(E)$ is a weak homotopy equivalence. The choice of a suitable ascending sequence S of finite dimensional subspaces of E determines an injection $i(S)$ of the infinite general linear group $GL(\infty)$ into $GL_c(E)$. This inclusion gives a weak homotopy equivalence of $GL(\infty)$ with $GL_F(E)$. Consequently

COROLLARY (PALAIS-ŠVARC). *The map* $i(S): GL(\infty) \to GL_c(E)$ *is a homotopy equivalence.*

2. **Layer structures and Fredholm maps.** A map α: $X \to E$ will be called locally finite dimensional (l.f.d.) if each point x in X has a neighbourhood N_x with $\alpha(N_x)$ contained in a finite dimensional subspace of E. If $T \in L(E, F)$ and U is an open subset of E a map f: $U \to F$ will be called an $L(T)$-map if $f - T$: $U \to F$ is l.f.d.

DEFINITION. If M is a C^p manifold, a 'C^p layer structure' on M, modelled on E, is a maximal C^p atlas $\{(U_i, \phi_i)\}_i$ for M, ϕ_i: $U_i \to E$, such that, when defined, $\phi_i \circ \phi_j^{-1}$ is an $L(I)$-map. A manifold with a layer structure will be called a layer manifold.

DEFINITION. A map between layer manifolds M and N will be called an $L(T)$-map if it is represented as an $L(T)$-map by the layer charts of M and N.

The collection of layer manifolds and L-maps is a category L which has most of the properties of the differentiable category, including the notions of embeddings, submanifolds, pull backs, and also sprays, exponential maps and tubular neighbourhoods.

THEOREM 2. *Let M be a C^p manifold.*

(i) *A $C^p\Phi_0$-map $f: M \to E$ induces a unique C^p layer structure $\{M, f\}$ on M, modelled on E, with respect to which f becomes an $L(I)$-map into E with its trivial layer structure.*

(ii) *If M is C^p-smooth, given a C^p layer structure M_1 on M, modelled on E, there is a $C^p\Phi_0$-map $f: M \to E$ such that $\{M, f\} = M_1$.*

(A differentiable map f between manifolds M and N is a 'Φ_n-map' if $Tf: TM \to TN$ is a Φ_n bundle map.)

REMARK. In (i) E may be replaced by any C^p layer manifold modelled on E; also there are corresponding results for Φ_n-maps. In order to obtain Φ_0-maps as in (ii) only an integrable $\mathrm{GL}_c(E)$-structure on M is needed. Thus such a weaker structure can be refined to a layer structure provided M is C^p-smooth. The category L can be considered as a tool to study the structures based on the various ideals of $L(E, E)$.

A layer structure on M, modelled on E, induces a reduction of TM to $GL_c(E)$. The following converse is proved using Proposition 1 and the exponential map of a spray.

THEOREM 3. *Let M be a C^p-smooth E-manifold, $p \geqq 3$. Then if TM admits a reduction to* $\mathrm{GL}_c(E)$ *there is a $C^p\Phi_0$-map $f: M \to E$.*

THEOREM 4. *Let M and N be C^p layer manifolds, modelled on E and F. Suppose that M is C^p-smooth and that N is paracompact. Then for any $T \in L(E, F)$, the C^p $L(T)$-maps $f: M \to N$ are dense in the fine topology of $C^0(M, N)$.*

REMARK. Since an $L(I)$-map is a Φ_0-map, Theorems 2, 3, and 4, give sufficient conditions for the C^0-approximation of continuous maps by Φ_0-maps.

The C^p homotopy classes through Φ_n-maps of $C^p\Phi_n$-maps $f: M \to N$ will be denoted by $\Phi_n[M, N]^p$.

THEOREM 5. *Let M and N be C^p-smooth E-manifolds. Then if TM is trivial and if TN admits a reduction to* $\mathrm{GL}_c(E)$ *there is a bijection,*

$$\Phi_0[M, N]^p \to [M, \Phi_0(E, E)] \times [M, N].$$

This is first proved for $N = E$, using Proposition 1 and its Corollary.

This part of the proof shows that any reduction of TM to $GL_c(E)$ is equivalent to an integrable reduction.

3. **Degree theory.** A Φ_n-map $f: M\to E$, $n\geqq 0$, induces, as in Proposition 1, a reduction $T\{M, f\}_k$, of TM to $GL_c(E\times R^n)$. Using this fact, the Palais-Švarc theorem, and Smale's extension of Sard's theorem, under certain conditions it is possible to give a degree theory for proper Φ_n-maps based on framed cobordism. The following example is meant only as a simple illustration of what can occur.

ILLUSTRATION. Let M be an n-connected separable C^{n+3}-manifold $n\geqq 0$, modelled on an infinite dimensional Hilbert space H. Then the proper homotopy classes of proper $C^{n+3}\Phi_n$-maps $f: M\to H$ with $T\{M, f\}_k$ trivial are in one-to-one correspondence with the set Σ_n' of the equivalence classes of the nth stable homotopy group of spheres under the relation $a\sim b$ iff $a=\pm b$.

The 'Pontrjagin-Thom construction' used here depends on Bessaga's theorem on the diffeomorphism of H with $H-\{0\}$, [1].

4. **Applications.** (a) *Embedding theorems.* Given a Φ_0-map $f: M\to E$ it is a simple matter to obtain a closed embedding $\bar{f}: M\to E\times F$. The following extension of J. McAlpin's embedding theorem for Hilbert manifolds follows from Theorem 2.

EMBEDDING THEOREM. *Let M be a separable C^p E-manifold, $p\geqq 3$. Suppose that TM admits an inverse modelled on a separable Banach space G. Then if $E\times G$ is C^p-smooth, there is a closed C^p embedding of M onto a closed (split) submanifold of $E\times G\times F$, for any infinite dimensional Banach space F. Also any continuous function of M into $E\times G\times F$ can be approximated in the fine C^0 topology by a one-to-one (split) immersion.*

(b) *Nonlinear elliptic equations.* The degree theory of §3 applies to proper Fredholm maps. A careful analysis of Smale's proof that Fredholm maps are locally proper, or equivalently of the proof of Theorem 2(i), shows

LEMMA. *Let $i: E_1\to E_0$ and $j: F_1\to F_0$ be continuous linear injections of Banach spaces E_1 and F_1 onto dense subspaces of Banach spaces E_0 and F_0. Suppose that $f_r: E_r\to F_r$, $r=0, 1$, are $C^p\Phi_n$-maps $p\geqq 1$, with $f_0\circ i=j\circ f_1$. Then, if i is a compact map, f_1 is proper on every closed bounded subset of E_1.*

This lemma can be applied to the maps induced on suitably chosen function spaces by sufficiently smooth nonlinear elliptic boundary value problems. The degree theory then gives direct proofs of results like those of F. Browder in [2].

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