A BASIS FOR THE LAWS OF PSL(2,5)

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1. Introduction. Although it is known that there is a finite basis for the laws of any finite group (Sheila Oates and M. B. Powell [6]), it is not in general an easy matter to find an explicit basis for the laws of a given finite group. Indeed, the set of laws given below is, as far as we know, the only explicit basis known for the laws of a finite non-abelian simple group.

Before writing down the basis we define the law $u_n$ introduced by L. G. Kovács and M. F. Newman [4]:

$$u_3 = \left[ (x_1^{-1} x_2)^{x_1 x_2}, (x_1^{-1} x_2^{-1})^{x_1} x_2, (x_2^{-1} x_3)^{x_1} x_3 \right]$$

and, for $n>3$,

$$u_n = \left[ u_{n-1}, (x_1^{-1} x_n)^{x_1 x_n}, \ldots, (x_{n-1} x_n)^{x_1 x_{n-1} x_n} \right].$$

**Theorem A.** The set of laws (1)–(7) given below is a basis for the laws of PSL(2, 5), the simple group of order 60.

1. $x^{20} = 1$
2. $\left[ (x^{10} y^{10})^6 [x^{10}, y^{10}]^2 \right]^6 = 1$
3. $\left[ ((x^6 y^{12})^8 (x^6 y^{12})^8 [x^6, y^6]^8]^6 \right]^6 = 1$
4. $[x^8, y^8]^{16} = 1$
5. $\left[ [x^6 y^{10}, y^{-10}] [y^{10}, x^8] \right]^{10} = 1$
6. $\left[ [y^{10} x^6 y^{-10}, x^{-6}] [y^{10}, x^6]^2 \right]^6 = 1$
7. $u_{41} = 1$.

It can be verified by direct calculation that PSL(2, 5) satisfies these laws, so it is sufficient to prove that the variety $\mathcal{B}$ defined by these laws is contained in the variety $\mathcal{B}_0$ generated by PSL(2, 5).

2. Notation. In notation and terminology we will follow the book of Hanna Neumann [5]; we will also assume familiarity with the results of Chapters 1 and 5 of this book.

3. Finite soluble groups in $\mathcal{B}$.

**Lemma 3.1.** Groups in $\mathcal{B}$ of prime-power order are elementary abelian.

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PROOF. Law (1) shows that groups whose orders are powers of 2 are elementary abelian, and that those whose orders are powers of 3 or 5 have exponent 3 or 5 respectively.

Law (2) shows that the commutator of two elements in the same 3-group has order 10, and hence, since it certainly has order 3, it is trivial, and the group is abelian.

Law (3) gives the corresponding result for 5-groups.

**Lemma 3.2.** An element of order $p$ which belongs to the normalizer of a $q$-subgroup of a group in $\mathfrak{B}$ belongs to its centralizer if $p$ and $q$ take the values 5 and 2, 5 and 3, or 3 and 5.

**Proof.** This follows from laws (4), (5) and (6). Consider, for instance, law (5). If $y$ is an element of a 3-subgroup and $x$ an element of order 5 in its normalizer, then the first factor in the law vanishes and we are left with $[y, x]^{10} = 1$; but certainly $[y, x]^{3} = 1$ and so $[y, x] = 1$, as required.

**Theorem 3.3.** A critical soluble group in $\mathfrak{B}$ belongs to $\mathfrak{s}_{2} \mathfrak{s}_{3}$, $\mathfrak{s}_{3} \mathfrak{s}_{2}$ or $\mathfrak{s}_{5} \mathfrak{a}_{2}$.

**Proof.** Let $G$ be a critical soluble group in $\mathfrak{B}$. Since all its Sylow subgroups are elementary abelian, by Theorem 1.2.6 of P. Hall and G. Higman [2], $G$ has $p$-length 1 for all primes $p$. Consider in particular, its upper 5-series. This has the form

$$1 = P_{0} \leq N_{0} \leq P_{1} \leq N_{1} = G,$$

where $N_{0}$ and $N_{1}/P_{1}$ are groups of order prime to 5 and $P_{1}/N_{0}$ is a 5-group. If $S$ is the Sylow 5-group of $P_{1}$ then, by Lemma 3.2, $S \leq C_{G}(N_{0})$ and so, since $P_{1} = N_{0}S, P_{1} = N_{0} \times S$. But $S$ is characteristic in $P_{1}$ and hence normal in $G$ and has trivial intersection with $N_{0}$ which is also normal in $G$. Since $G$ is critical we must have $N_{0} = 1$ or $S = 1$.

(i) $N_{0} = 1$. Then $P_{1}$ is a normal Sylow 5-subgroup of $G$. If $|G|$ were divisible by 3, then, by Lemma 3.2, there would be elements outside $P_{1}$ which centralized $P_{1}$, and this is impossible by Lemma 1.2.3 of [2]. Thus $G/P_{1}$ has order a power of 2, and hence, since both $G/P_{1}$ and $P_{1}$ are elementary abelian, $G \subseteq \mathfrak{s}_{2} \mathfrak{s}_{3}$.

(ii) $P_{1} = 1$. Then the order of $G$ is divisible only by powers of 2 and 3, and, since $G$ is critical, it cannot possess both a nontrivial normal 2-subgroup and a nontrivial normal 3-subgroup. If $G$ has no non-trivial normal 2-subgroup then its upper 3-series is

$$1 = P_{0} = N_{0} < P_{1} \leq N_{1} = G$$

where $P_{1}$ is a 3-group and $G/N_{1}$ a 2-group. Hence $G \subseteq \mathfrak{s}_{2} \mathfrak{s}_{3}$. An analog-
gous argument shows that if \( G \) has no nontrivial normal 3-subgroup then \( G \in \mathcal{V}_3 \).

**Corollary 3.4.** Finite soluble groups in \( \mathcal{B} \) belong to \( \mathcal{B}_0 \).

**Proof.** By Corollary 4.2.4 of P. J. Cossey [1], \( \mathcal{V}_{31} \), \( \mathcal{V}_{32} \) and \( \mathcal{V}_{33} \) are generated, respectively, by the dihedral groups of orders 10 and 6, and the alternating group on 4 letters. Since these are all subgroups of PSL(2, 5) the varieties \( \mathcal{V}_3, \mathcal{V}_3^2 \) and \( \mathcal{V}_3^3 \) are subvarieties of \( \mathcal{B}_0 \). Thus all critical soluble groups in \( \mathcal{B} \) are in \( \mathcal{B}_0 \), and, by induction on the order, we see that any finite soluble group in \( \mathcal{B} \) is in \( \mathcal{B}_0 \).

4. Finite nonsoluble groups in \( \mathcal{B} \).

**Lemma 4.1.** The only nonabelian simple group in \( \mathcal{B} \) is PSL(2, 5).

**Proof.** By (4.4) of L. G. Kovács and M. F. Newman [4], the law \( u_6 = 1 \) implies that, for any group in \( \mathcal{B} \), the index of the centralizer of a chief factor cannot exceed 60, and hence any nonabelian simple group in \( \mathcal{B} \) has order \( \leq 60 \). Since PSL(2, 5) is the only simple group with this property the result follows.

**Theorem 4.2.** Every finite group in \( \mathcal{B} \) is of the form

\[ A_1 \times \cdots \times A_r \times S \]

where \( A_i \cong \text{PSL}(2, 5) \) (\( i = 1, \cdots , r \)) and \( S \) is soluble.

**Proof.** Suppose not, and let \( G \) be a minimal counterexample, then \( G \) is certainly critical and not soluble. We have two cases to consider, according as the monolith \( \sigma G \) of \( G \) is abelian or nonabelian.

(i) \( \sigma G \) nonabelian. Then \( \sigma G \) is a direct product of groups isomorphic to \( \text{PSL}(2, 5) \) and the centralizer of \( \sigma G \) in \( G \) is 1. Hence \( G \) is an automorphism group of \( \sigma G \) and so has the form

\[ 1 \leq K \leq G \]

where \( K \) is a direct product of groups isomorphic to either \( \text{PSL}(2, 5) \) or \( S_5 \) (the symmetric group on 5 letters) and \( G/K \) acts as a transitive permutation group on the direct factors of \( K \). Because of the exponent law, \( S_5 \) cannot occur, and so \( K \) is a direct product of groups isomorphic to \( \text{PSL}(2, 5) \) and it follows that \( K = \sigma G \). If \( G/K \neq 1 \), there is an element of prime power order acting nontrivially on \( \sigma G \) and so \( G \) would not have abelian Sylow subgroups. We deduce that \( G = K \cong \text{PSL}(2, 5) \).

(ii) \( \sigma G \) abelian. By the minimality of \( G \), \( G/\sigma(G) \) is a direct product in which at least one factor isomorphic to \( \text{PSL}(2, 5) \) occurs (since \( G \)
is not soluble). Suppose \( G/\sigma(G) = H_1/\sigma(G) \times H_2/\sigma(G) \) where \( H_1/\sigma(G) \cong \text{PSL}(2, 5) \) and \( H_2 > \sigma(G) \). Then \( H_1 < G \) and so is of the form \( K_1 \times \sigma(G) \) where \( K_1 \cong \text{PSL}(2, 5) \). Thus \( K_1 \) induces automorphisms of \( H_2 \) which are trivial on \( \sigma(G) \) and on \( H_2/\sigma(G) \). But any two such automorphisms commute, and, since the only abelian factor group of \( K_1 \) is the trivial group, it follows that \( K_1 \) centralizes \( H_2 \). Thus \( G = K_1 \times H_2 \) is not critical. Hence we must have \( H_1 = G \), i.e. \( G/\sigma(G) \cong \text{PSL}(2, 5) \).

Now \( \sigma(G) \) is a \( p \)-group for \( p \in \{2, 3, 5\} \) and, if it were not central in \( G \), then \( G \) would have non-abelian \( p \)-subgroups. But, if \( \sigma(G) \) were central in \( G \), then, since \( G' \neq 1 \), \( \sigma(G) \leq G' \cap Z(G) \) which is 1 for a group with abelian Sylow subgroups by 3.2 of B. Huppert [3]; so again we have a contradiction.

**Corollary 4.3.** Finite groups in \( \mathfrak{B} \) are in \( \mathfrak{B}_0 \).

5. **Proof of Theorem A.** Since we have shown that finite groups in \( \mathfrak{B} \) are in \( \mathfrak{B}_0 \), the proof of Theorem A will be complete if we can show that \( \mathfrak{B} \) is locally finite, since a variety is determined by its finitely generated groups. Now, the finite groups in \( \mathfrak{B} \) on a fixed number of generators have bounded order, and so, if \( \mathfrak{B} \) were not locally finite it would contain a nonabelian infinite simple group, contradicting Lemma 4.1.

6. **Remarks.** We have avoided the use of (7) whenever possible, for the reason that we would like to delete it or at least replace it by a set of laws involving only a small number of variables. This however we have been unable to do. We have used it to show that \( \text{PSL}(2, 5) \) is the only nonabelian finite simple group in \( \mathfrak{B} \); this could have been avoided by appealing to the (unpublished) classification of simple groups all of whose Sylow subgroups are abelian. This still leaves the problem of local finiteness of \( \mathfrak{B} \). Using arguments similar to those of §3.3 of [6] we can show that there is a set of 5 variable laws which imply local finiteness; we have not been able to find such a set explicitly, however.

**References**

CONTINUITY OF THE VARISOLVENT CHEBYSHEV OPERATOR

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In this note we show that the Chebyshev operator $T$ is continuous at all functions whose best approximations are of maximum degree. Let $F$ be an approximating function unisolvent of variable degree on an interval $[a, \beta]$ and let the maximum degree of $F$ be $n$. Let $P$ be the parameter space of $F$. All functions considered will be continuous and for such functions we define the norm
\[ \|g\| = \max\{ |g(x)| : \alpha \leq x \leq \beta \} . \]
The Chebyshev problem is, for a given continuous function $f$, to find an element $T(f) = F(A^*, \cdot )$, $A^* \in P$, for which
\[ \rho(f) = \inf \{ \|f - F(A, \cdot )\| : A \in P \} \]
is attained. Such an element $T(f)$ is called a best Chebyshev approximation to $f$ on $[\alpha, \beta]$. $T(f)$ can fail to exist, but is unique and characterized by alternation if it exists. Definitions and theory are given in [1].

**Lemma 1.** Let $F(A, \cdot )$ be the best approximation to $f$ and $F$ have degree $n$ at $A$. Let $x_0, \cdots, x_n$ be an ordered set of points on which $f - F(A, \cdot )$ alternates $n$ times. If $\|f - g\| < \delta$ and $\|g - F(B, \cdot )\| \leq \rho(g) + \delta$ then
\[ (1) \quad (-1)^i [F(B, x_i) - F(A, x_i)] \quad \text{sgn}(f(x_0) - F(A, x_0)) \geq -3\delta, \]
for $i = 0, \cdots, n$.

The lemma can be obtained using arguments similar to those of Rice [2, p. 63].

**Lemma 2.** Let $F$ be of degree $n$ (maximal) at $A$ then for given $\delta > 0$