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CONTINUITY OF THE VARISOLVENT CHEBYSHEV OPERATOR

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In this note we show that the Chebyshev operator T is continuous at all functions whose best approximations are of maximum degree. Let F be an approximating function unisolvent of variable degree on an interval $[\alpha, \beta]$ and let the maximum degree of F be n. Let P be the parameter space of F. All functions considered will be continuous and for such functions we define the norm

$$\|g\| = \max\{ | g(x) | : \alpha \leqq x \leqq \beta\}.$$

The Chebyshev problem is, for a given continuous function f, to find an element $T(f) = F(A^*, \cdot)$, $A^* \in P$, for which

$$\rho(f) = \inf\{\|f - F(A,\cdot)\| : A \in P\}$$

is attained. Such an element $T(f)$ is called a best Chebyshev approximation to f on $[\alpha, \beta]$. $T(f)$ can fail to exist, but is unique and characterized by alternation if it exists. Definitions and theory are given in [1].

LEMMA 1. *Let $F(A, \cdot)$ be the best approximation to f and F have degree n at A. Let $x_0, \cdots, x_n$ be an ordered set of points on which $f - F(A, \cdot)$ alternates n times. If $\|f-g\| < \delta$ and $\|g - F(B, \cdot)\| \leqq \rho(g) + \delta$ then*

$$(1) \quad (-1)^i[F(B, x_i) - F(A, x_i)] \operatorname{sgn}(f(x_0) - F(A, x_0)) \geqq -3\delta, \qquad i = 0, \cdots, n.$$

The lemma can be obtained using arguments similar to those of Rice [2, p. 63].

LEMMA 2. *Let F be of degree n (maximal) at A then for given $\delta > 0$*

there exists $\eta(\delta)$ *such that* $\|F(A,\cdot)-F(B,\cdot)\|<\eta(\delta)$ *if* (1) *holds and* $\eta(\delta)\to 0$ *as* $\delta\to 0$.

The lemma is proven by arguments analogous to those of Tornheim cited after the next lemma.

LEMMA 3. *Let* F *be unisolvent of degree* m *at* A_k, $k=0, 1, \cdots$ *and let* $\{F(A_k,\cdot)\}$ *converge pointwise to* $F(A_0,\cdot)$ *on* m *distinct points then* $\{F(A_k,\cdot)\}$ *converges uniformly to* $F(A_0,\cdot)$.

This result is a generalization of a result of Tornheim [**2**, pp. 72–73], [**3**, pp. 460–462] and is proven in the same way.

THEOREM. *Let* F *be unisolvent of variable degree. Let* f *have a best approximation* $F(A,\cdot)$ *and* F *be of degree* n *(maximal) at* A. *There exists* $\delta>0$ *such that* $\|f-g\|<\delta$ *implies that* g *has a best approximation. If* $\{f_k\}$ *converges uniformly to* f *then* $\{T(f_k)\}$ *converges uniformly to* $T(f)$.

PROOF. Let $x_0, \cdots, x_n$ be as in Lemma 1. By definition of solvency of degree n at A there exists $\gamma>0$ such that if $|y_k-F(A,x_k)|<\gamma$, $k=1,\cdots,n$, then there exists a parameter B satisfying

$$F(B,x_k)=y_k, \qquad k=1,\cdots,n. \tag{2}$$

Using property Z and maximality of n, it is easily seen that F is unisolvent of degree n at any such B, and hence B is completely determined by (2). Choose δ such that $\eta(\delta)<\gamma/2$ then by Lemmas 1 and 2, if $\|f-g\|<\delta$ and $\|g-F(B,\cdot)\|<\rho(g)+\delta$, we have $\|F(A,\cdot)-F(B,\cdot)\|<\gamma/2$. Now let $\|g-F(B_k,\cdot)\|$ be a decreasing sequence with limit $\rho(g)$, then for all k sufficiently large, $\|F(A,\cdot)-F(B_k,\cdot)\|<\gamma/2$. The n-tuples of values at the points $x_1,\cdots,x_n$ of the approximants $F(B_k,\cdot)$ form therefore a bounded sequence with subsequence converging to an accumulation point $(y_1,\cdots,y_n)$, which determines a parameter B at which F is unisolvent of degree n. Using Lemma 3 we can show that for all $x\in[\alpha,\beta]$, $|f(x)-F(B,x)|\leqq\rho(g)$ and so $F(B,\cdot)$ is a best approximation to g. The first part of the theorem is proven. Now let $\{f_k\}$ converge uniformly to f, then for all k sufficiently large, $T(f_k)$ exists. From Lemmas 1 and 2 it follows immediately that $\|T(f)-T(f_k)\|$ converges to zero. The theorem is proven. From the arguments involving n-tuples we obtain

COROLLARY. *Let* F *be unisolvent of variable degree, then the set of approximants of maximum degree is locally compact.*

In developing the paper, no assumptions were made concerning the existence of $T(f)$. In case a unique best approximation exists to every continuous function, it is easily shown that if f is an approximant, $\{f_k\}$ converging uniformly to f implies that $\{T(f_k)\}$ converges uniformly to f, and the operator T is continuous at every continuous function which is an approximant or has a best approximation of maximum degree. In the case of approximation by generalized rational functions it has been shown by Cheney and Loeb [4] that T is continuous at no other continuous functions.

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