GERŠGORIN THEOREMS BY HOUSEHOLDER'S PROOF

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0. **The method.** Given an $m\times m$ matrix $A=[a_{ij}]$ of complex numbers, S. Geršgorin [4] proved that every proper value λ lies in the union of the m disks D_i, where $D_i\equiv\{\lambda \mid |\lambda-a_{ii}|<R_i,\ R_i=\sum_{j\neq i}|a_{ij}|\}$. Generalizations of this theorem have appeared in several papers, see for example [1], [3], [5], [7], [8], and a convenient summary in [6]. The theorem is derivable from the following (older) result, if we set $B=A-\lambda I$.

THEOREM 1. *Let $B=[b_{ij}]$ be a matrix of complex numbers. If B is not invertible, then for some i we must have $|b_{ii}|\leqq\sum_{j\neq i}|b_{ij}|=R_i$.*

COROLLARY. $\forall_i\{|b_{ii}|>R_i\}\Rightarrow B$ *is invertible.*

This is the contrapositive of Theorem 1. To prove Theorem 1, find $x=\{x_1, x_2, \cdots, x_n\}$ so that $Bx=0$; choose i so that $x_i\neq 0$ and $\forall_j\{|x_i|\geqq|x_j|\}$. Then $|b_{ii}|\leqq\sum|b_{ij}|\cdot|x_j/x_i|\leqq R_i$.

Householder [5, p. 66] looks at the theorem from a different point of view. He writes $B=D-C$, where D is the diagonal part of B, i.e. $D=[d_{ij}]$, $d_{ij}=\delta_j^i\cdot b_{ij}$, and C has zero diagonal. If $\forall_i\{b_{ii}\neq 0\}$, then $B=D(I-D^{-1}C)$. The condition $\|D^{-1}C\|<1$ guarantees that B be invertible. The corollary follows on applying this condition and using the row-sum norm.

1. **A new result.** In the preceding paragraph, a known result was recovered by Householder's method. This does not demonstrate the full power of the method. In this section, we obtain a new result by the same method. (This result can be obtained also by other methods; see [2].)

DEFINITION. The notation

$$B\begin{pmatrix}1 & \cdots & n\\ 1 & \cdots & n\end{pmatrix}$$

means the minor matrix obtained from the large matrix B by retaining only rows $1\cdots n$ and columns $1\cdots n$. The notation

$$B\begin{pmatrix}1\cdots n\\ \{1\cdots n\}\backslash t, j\end{pmatrix}$$

means the minor matrix based on rows $1 \cdots n$ and columns $1 \cdots n$, but with column t omitted and column j ($j > n$) appended.

LEMMA. *Let c_{tk} be the t, k element of*

$$W = B\begin{pmatrix} 1 & \cdots & n \\ 1 & \cdots & n \end{pmatrix}^{-1}.$$

Then

$$\left|\det\left\{WB\begin{pmatrix} 1 \cdots n \\ \{1 \cdots n\}\backslash t, j \end{pmatrix}\right\}\right| = \left|\sum c_{tk} b_{kj}\right|.$$

PROOF. The matrix product Q on the left side of the lemma is equal to the identity matrix except in the tth column, which is replaced by the jth column as shown. The determinant of Q is therefore equal to the t, t element of Q, i.e. the inner product of the tth row of W by the jth column of B.

THEOREM 2. *Let B be an $m\times m$ matrix of complex numbers; let $S(1)$, $S(2), \cdots$ be a partitioning of $\{1 \cdots m\}$ into disjoint sets. Let*

$$V(r) = \begin{pmatrix} S(r) \\ S(r) \end{pmatrix}$$

be the (principal) submatrix of B on the rows and columns with indices in $S(r)$. Let

$$U(r, j, t) = A\begin{pmatrix} S(r) \\ S(r)\backslash j, t \end{pmatrix}$$

be the submatrix of B that uses rows with indices in $S(r)$, and columns with indices from the same set, but with the column of index j deleted and the column of index t appended.

The matrix B is nonsingular if the following m inequalities hold among certain minor determinants of B:

$$\forall_{j\in S(r)}\ \forall_r \left\{ \left|\det V(r)\right| > \sum_{t\in S(r)} \left|\det U(r, j, t)\right| \right\}.$$

REMARK. If $S(i) = \{i\}$, this theorem reduces to the Geršgorin corollary.

PROOF. We write $B = D - C = D(I - D^{-1}C)$ as before, but interpret D as the block diagonal $V(1) \dotplus V(2) \dotplus \cdots$ of B. If we apply the lemma (read from right to left) to the matrix $D^{-1}C$, and use row-sum norm in the condition $\|D^{-1}C\| < 1$, Theorem 2 follows.

COROLLARY. *Every proper value of the matrix A lies in one or another of the m loci*

$$\left|\det\begin{pmatrix} a_{r,r}-\lambda, & a_{r,r+1} \\ a_{r+1,r}, & a_{r+1,r+1}-\lambda \end{pmatrix}\right| \leqq \sum_{t\neq r,r+1}'' \left|\det\begin{pmatrix} a_{r,r}-\lambda & a_{r,t} \\ a_{r+1,r} & a_{r+1,t} \end{pmatrix}\right|,$$

$$\left|\det\begin{pmatrix} a_{r,r}-\lambda, & a_{r,r+1} \\ a_{r+1,r}, & a_{r+1,r+1}-\lambda \end{pmatrix}\right| \leqq \sum'' \left|\det\begin{pmatrix} a_{r,t} & a_{r,r+1} \\ a_{r+1,t} & a_{r+1,r+1}-\lambda \end{pmatrix}\right|,$$

$r=1, 3, 5, \cdots, m-1$. (*If m is odd, the last value of r is* $m-2$, *and the disk* $|a_{mm}-\lambda| \leqq R_m$ *must be appended.*)

This corollary has been used in numerical analysis, in a case in which complex numbers are replaced by 2×2 matrices.

REFERENCES

1. A. Brauer, *Limits for the characteristic roots of a matrix*, Duke Math. J. **19** (1952), 75–91.

2. J. L. Brenner, *New root-location theorems for partitioned matrices*, Summary Report, #748, MRC, Madison, Wis., 1967; SIAM J. Appl. Math. **16**, No. 3 (1968).

3. D. G. Feingold and R. S. Varga, *Block diagonally dominant matrices and generalizations of the Geršgorin circle theorem*, Pacific J. Math. **12** (1962), 1241–1250.

4. S. Geršgorin, *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk SSSR **7** (1931), 749–754.

5. A. S. Householder, *The theory of matrices in numerical analysis*, Blaisdell, New York, 1964.

6. M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, Mass., 1964.

7. A. M. Ostrowski, *Sur les conditions générales pour la regularité des matrices*, Rend. Mat. e Appl. **10** (1951), 1–13.

8. O. Taussky, *A recurring theorem on determinants*, Amer. Math. Monthly **56** (1948), 672–676.

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