A FIXED POINT THEOREM FOR SET VALUED MAPPINGS[1]

BY JACK T. MARKIN

Communicated by C. B. Morrey, Jr., February 5, 1968

Let H be a real Hilbert space with closed unit ball B and let $K(H)$ denote the family of nonempty compact convex subsets of H supplied with the Hausdorff metric D generated by the norm of H. A mapping ϕ: $H \to K(H)$ is *contractive* if for any pair $x, y \in H$, $D(\phi(x), \phi(y)) \leqq D(x, y)$. If $x \in \phi(x)$, then x is a fixed point of ϕ.

In this paper we shall prove the following fixed point theorem for set valued contractions, which is an extension of a theorem of Browder [1].

THEOREM 1. *Let* ϕ: $H \to K(H)$ *be a contractive mapping such that* $\phi(x) \subset B$ *for every* $x \in B$. *Then* ϕ *has a fixed point in* B.

The proof relies on a generalization of the concept of monotone mappings of H into H to mappings of H into $K(H)$, and also depends on Theorem 2 which we state without proof.

THEOREM 2. *Assume that* X *is a complete bounded metric space and that* ϕ *maps* X *into the family of nonempty closed subsets of* X. *If there is a* $k \in [0, 1)$ *such that for any pair* $x, y \in X$, $D(\phi(x), \phi(y)) \leqq kD(x, y)$, *then* ϕ *has a fixed point.* (*Here* D *is the Hausdorff metric generated by the metric of* X.)

A mapping G of H into the family of nonempty subsets of H is *monotone* if given $u, v \in H$ and $\bar{u} \in G(u)$ there is a $\bar{v} \in G(v)$ such that $(\bar{u} - \bar{v}, u - v) \geqq 0$.

LEMMA 1. *Let* G: $H \to K(H)$ *be a continuous monotone map, and assume that for some pair* $v, \bar{v} \in H$ *and every* $u \in H$ *there is a* $\bar{u} \in G(u)$ *such that* $(\bar{v} - \bar{u}, v - u) \geqq 0$. *Then,* $\bar{v} \in G(v)$.

PROOF. Suppose $\bar{v} \notin G(v)$. By weak compactness, there is a $w \in H$ such that $(\bar{v}, w) < (x, w)$ for every $x \in G(v)$. Let $v_n = v - (1/n)w$; since G is monotone there exists a $\bar{v}_n \in G(v_n)$ such that $(\bar{v} - \bar{v}_n, v - v_n) \geqq 0$ for all n. Therefore $(\bar{v}, w) \geqq (\bar{v}_n, w)$. Since $D(G(v_n), G(v))$ tends to 0 as $n \to \infty$ by continuity, we may assume that $\{\bar{v}_n\}$ tends weakly to a point $\bar{x}$ and that there is a sequence $\{z_n\}$, $z_n \in G(v)$, such that

[1] This work was supported by the U. S. Army Research Office-Durham, contract DA-31-124-ARO-D-265.

$\lim D(\bar{v}_n, z_n) = 0$ and $\{z_n\}$ converges weakly to a point $z_0 \in G(v)$. We assert that $\bar{x} \in G(v)$ so that $(\bar{v}, w) \geqq (\bar{x}, w)$, which is absurd. Indeed, if $\bar{x} \notin G(v)$ there is a $\bar{z} \in H$ such that $(\bar{x}, \bar{z}) < (z_0, \bar{z})$ so that

$$0 < (z_0 - \bar{x}, \bar{z}) = (z_0 - z_n, \bar{z}) + (z_n - \bar{v}_n, \bar{z}) + (\bar{v}_n - \bar{x}, \bar{z}),$$

which is absurd (because the right side converges to 0 as $n \to \infty$).

LEMMA 2. *If G: $H \to K(H)$ is a continuous monotone mapping then $G(B)$ is closed in the norm topology.*

PROOF. If v_0 is a limit point of $G(B)$ there is a sequence $\{\bar{u}_j\}$ in $G(B)$ such that $\bar{u}_j \in G(u_j)$, $\bar{u}_j \to v_0$ and $u_j \in B$. Since B is weakly compact we may assume that u_j converges weakly to $u_0 \in B$.

If for every $u \in H$ there is a $\bar{u} \in G(u)$ such that $(\bar{u} - v_0, u - u_0) \geqq 0$, then Lemma 1 implies that $v_0 \in G(u_0)$ and the proof is complete. If not, there is a $v \in H$ and an $\epsilon > 0$ such that

$$(1) \qquad (\bar{v} - v_0, v - u_0) \leqq -\epsilon < 0$$

for all $\bar{v} \in G(v)$. We shall show that this leads to a contradiction.

Since G is monotone there is a $\bar{v}_j \in G(v)$ satisfying $(\bar{v}_j - \bar{u}_j, v - u_j) \geqq 0$ for each positive integer j. By the compactness of $G(v)$ we may assume that $\bar{v}_j \to w \in G(v)$. Therefore, $(\bar{v}_j - \bar{u}_j, v - u_j) \to (w - v_0, v - u_0) \geqq 0$, which contradicts (1).

Given any $u, v \in H$ and $\bar{u} \in \phi(u)$ the compactness of $\phi(v)$ guarantees a point $\bar{v} \in \phi(v)$ such that $D(\bar{u}, \bar{v}) \leqq D(\phi(u), \phi(v))$. Therefore, $((u - \bar{u}) - (v - \bar{v}), u - v) \geqq (D(u, v))^2 - D(\bar{u}, \bar{v}) \cdot D(u, v) \geqq 0$, and hence $I - \phi$ is a monotone mapping. Clearly, $I - \phi$ is continuous as a mapping of H into $K(H)$.

If we show that 0 is in the closure of $(I - \phi)(B)$, then Theorem 1 will follow from the lemmas.

Let $\{k_i\}$ be a sequence in $(0, 1)$ which converges to 1. For each i, $k_i\phi$ maps B into the family of nonempty closed convex subsets of B, and satisfies the hypotheses of Theorem 2. Therefore, for each i there is a fixed point $u_{k_i} \in k_i\phi(u_{k_i})$. Clearly, $u_{k_i} = k_i v_{k_i}$ for some $v_{k_i} \in \phi(u_{k_i})$ and hence

$$\inf_{y \in \phi(u_{k_i})} D(u_{k_i}, y) \leqq D(u_{k_i}, v_{k_i}) \leqq 1 - k_i \to 0 \quad \text{as } i \to \infty.$$

REFERENCE

1. F. E. Browder, *Fixed point theorems for non-compact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1272–1276.

UNIVERSITY OF SOUTHERN CALIFORNIA