NONLINEAR EIGENVALUE PROBLEMS AND GALERKIN APPROXIMATIONS

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Let $X$ be a reflexive Banach space, $T$ and $S$ two mappings of $X$ into its conjugate space $X^*$. We denote the pairing between $w$ in $X^*$ and $u$ in $X$ by $(w, u)$, and weak convergence (in either $X$ or $X^*$) by $\rightharpoonup$, strong convergence (in either $X$ or $X^*$) by $\to$.

By an eigenvalue problem for the pair $(T, S)$, we mean the problem of finding an element $u$ in $X$ and a real number $\lambda$ such that

$$T(u) = \lambda S(u),$$

with $u$ possibly satisfying additional normalization conditions. It is our purpose in the present note to describe a way of applying a method of Galerkin type to such problems which works in particular for nonlinear elliptic boundary value problems of variational type. We obtain from it a general theorem on the existence of normalized eigenfunctions for the latter problem, and in the case of $T$ and $S$ odd operators, we obtain also an extremely general form of a theory of Lusternik-Schnirelman type guaranteeing the existence of infinitely many distinct normalized eigenfunctions.

We consider first some restrictions that may be placed on the nonlinear operator $T$.

**Definition 1.** $T$ is said to satisfy condition $(S)$ if for any sequence $\{u_j\}$ in $X$ with $u_j \rightharpoonup u$ in $X$ and $(T(u_j) - T(u), u_j - u) \to 0$, we have $u_j \to u$ in $X$.

**Definition 2.** $T$ is said to satisfy condition $(S)_0$ if for each sequence $\{u_j\}$ in $X$ with $u_j \rightharpoonup u$ in $X$, $T(u_j) \to z$ in $X^*$, and $(T(u_j), u_j) \to (z, u)$, we have $u_j \to u$ in $X$.

**Lemma 1.** (a) If $T$ satisfies condition $(S)$, it satisfies condition $(S)_0$.

(b) If $T$ is continuous and satisfies condition $(S)_0$, and if $K$ is any compact set of $X^*$, $B$ any bounded closed set of $X$, then $T^{-1}(K) \cap B$ is compact.

(c) If $T$ is continuous and satisfies condition $(S)_0$, then the image under $T$ of any bounded closed set $B$ of $X$ is closed in $X^*$.

**Proof of Lemma 1.** Proof of (a). Suppose $u_j \rightharpoonup u$, $T(u_j) \to z$, and $(T(u_j), u_j) \to (z, u)$. Then
\[(T(u_j) - T(u), u_j - u) = (T(u_j), u_j) - (T(u), u) - (T(u), u_j - u) \]
\[\rightarrow (z, u) - (z, u) - 0 = 0.\]

Hence by the condition \((S)\), \(u_j \rightarrow u\).

**Proof of (b).** Let \(\{u_j\}\) be a sequence in \(T^{-1}(K) \cap B\). By passing to a subsequence, we may assume that \(u_j \rightarrow u\) in \(X\), \(T(u_j) \rightarrow z\) in \(K\). Hence \((T(u_j), u_j) \rightarrow (z, u)\) and, by condition \((S)_0\), \(u_j \rightarrow u\). Hence \(u \in B\), and by the continuity of \(T\), \(T(u) = z\), i.e. \(u \in T^{-1}(K) \cap B\). Q.E.D.

**Proof of (c).** The conclusion of (b) implies that \(T\) is a proper continuous map of \(B\) into \(X^*\). Hence it is a closed map of \(B\) into \(X^*\) and \(T(B)\) is closed in \(X^*\). Q.E.D.

We now give our principal methodological result.

**Theorem 1.** Let \(X\) be a separable reflexive Banach space, \(T\) and \(S\) two continuous bounded mappings of \(X\) into \(X^*\) with \(T\) satisfying condition \((S)_0\) and \(S\) a compact map of \(X\) into \(X^*\). Let \(\{X_n\}\) be an increasing sequence of finite dimensional subspaces of \(X\) whose union is dense in \(X\), \(B\) a closed bounded subset of \(X\). Suppose that for each \(n\), there exists an element \(u_n\) of \(B \cap X_n\) with the property that

\[j_n^*T(u_n) = \lambda_n j_n^*S(u_n),\]

where \(j_n\) is the injection mapping of \(X_n\) into \(X\), and \(j_n^*\) is the dual projection of \(X^*\) onto \(X_n^*\). Suppose that \(|\lambda_n|\) is uniformly bounded.

Then there exists an eigenfunction \(u\) of the pair \((T, S)\) in \(B\), i.e. \(T(u) = \lambda S(u)\), and for any weakly convergent subsequence \(u_n(k) \rightarrow u\) of the sequence \(\{u_n\}\), \(u\) is such an eigenfunction and \(u_n(k) \rightarrow u\).

**Proof of Theorem 1.** Since \(B\) is bounded and \(X\) is reflexive, the sequence \(\{u_n\}\) has a weakly convergent subsequence. We may replace the original sequence by this subsequence and assume that \(u_n \rightarrow u\).

It suffices to show that \(\{u_n\}\) has a strongly convergent subsequence and that \(u\) is an eigenfunction of the pair \((T, S)\). Since \(|\lambda_n|\) is uniformly bounded, we may assume for our original sequence (again by passing to an infinite subsequence) that \(\lambda_n \rightarrow \lambda\), and since \(S\) is compact, that \(S(u_n) \rightarrow w\) in \(X^*\).

Let \(v\) be any element of \(V_m\) for some \(m\), and consider \(n \geq m\). Then,

\[\langle T(u_n), v \rangle = \langle T(u_n, j_n^*v) \rangle = \langle j_n^*T(u_n), v \rangle = \lambda_n \langle j_n^*S(u_n), v \rangle = \lambda_n \langle S(u_n), v \rangle.\]

Hence

\[(T(u_n), v) \rightarrow \lambda \langle w, v \rangle, (n \rightarrow + \infty).\]

Since this is true for each \(v\) in the dense union of the spaces \(V_m\) and since the sequence \(\{T(u_n)\}\) is bounded, it follows that \(T(u_n) \rightarrow \lambda w\).
On the other hand, by the same argument,

\[(T(u_n), u_n) = \lambda_n(S(u_n), w) \to \lambda(w, v).\]

Applying the condition \((S) \circ\) for \(T\), we see that \(u_n \to u\). Since \(T\) and \(S\) are continuous, \(T(u_n) \to T(u)\), \(S(u_n) \to w\). Hence

\[T(u) = \lim_n T(u_n) = \lambda w = -S(u).\]

Q.E.D.

The special interest of the conditions \((S)\) and \((S) \circ\) is that they are satisfied by quasi-linear elliptic differential operators in generalized divergence form under extremely weak hypotheses on the operators.

**Theorem 2.** Let \(\Omega\) be a bounded open set in \(\mathbb{R}^n\) for which the Sobolev Imbedding Theorem is valid, \(A\) and \(B\) two differential operators on \(\Omega\) of the form

\[A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, Du, \ldots, D^m u),\]

\[B(u) = \sum_{|\beta| \leq m-1} (-1)^{|\beta|} D^\beta B_\beta(x, u, \ldots, D^m u).\]

For each \(\alpha\) and \(\beta\), let \(A_\alpha(x, \xi)\) and \(B_\beta(x, \xi)\) be continuous in \(x\) and Lebesgue measurable in \(\xi\). Suppose that for a given exponent \(p\) with \(1 < p < +\infty\), \(V\) is a closed subspace of the Sobolev space \(W^{m,p}(\Omega)\) and for \(u\) and \(v\) in \(V\), we set

\[a(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(x, u, Du, \ldots, D^m u), D^\alpha v),\]

\[b(u, v) = \sum_{|\beta| \leq m-1} (-1)^{|\beta|} D^{\beta} B_\beta(x, u, \ldots, D^m u),\]

(with \((w, v) = \int_\Omega w v\)). Suppose that the following three conditions are satisfied:

1. There exists a constant \(c_0\) and functions \(c_\alpha\) in \(L^p(\Omega)\) such that

\[|A_\alpha(x, \xi)| \leq c_\alpha(x) + c_0 \sum_{|\phi| = m} |\xi_\phi|^{p-1} + \sum_{|\phi| \leq m-1} |\xi_\phi|^{q_\phi},\]

\[|B_\beta(x, \xi)| \leq c_\beta(x) + c_0 \sum_{|\phi| = m} |\xi_\phi|^{q_\phi},\]

where

\[q_\phi < p_\phi q_\phi^{-1},\]

\[q_\alpha = \max(1, np(n p - n + p(m - |\alpha|))^{-1}),\]

\[p_\phi^{-1} = \max(0, np(n - p(m - |\phi|))^{-1}).\]

2. For \(\psi = \{\psi_\beta: |\beta| \leq m-1\}, \xi = \{\xi_\alpha: |\alpha| = m\}\), set \(A_\alpha(x, \psi, \xi)\)
\[ \mathcal{A}(x, \xi) \] where \( \xi = [\psi, \zeta] \). Then for every \( x \) in \( \Omega \), \( \psi, \zeta \) and \( \zeta' \) with \( \zeta \neq \zeta' \),

\[ \sum_{|a|=m} [\mathcal{A}_a(x, \psi, \zeta) - \mathcal{A}_a(x, \psi, \zeta')] \langle \zeta_a - \zeta'_a \rangle > 0. \]

(3) There exist positive constants \( c_1 \) and \( c_2 \) such that

\[ \sum_{|a|=m} \mathcal{A}_a(x, \xi) \xi_a \geq c_1 |\xi|^p - c_2. \]

Then:

(a) The form \( a(u, v) \) is well defined for all \( u \) and \( v \) in \( V \) and there exists an unique element \( T(u) \) in \( V^* \) such that \( a(u, v) = (T(u), v) \) for all \( v \) in \( V \) and a given element \( u \) in \( V \). Similarly, \( b(u, v) \) is well defined for \( u \) and \( v \) and \( b(u, v) = (S(u), v) \) for all \( v \) in \( V \) and a given \( u \) in \( V \), when \( S(u) \in V^* \).

(b) \( T \) is a bounded continuous mapping of \( V \) into \( V^* \) which satisfies condition \((S)\).

(c) \( S \) is a compact mapping of \( V \) into \( V^* \).

The proof of Theorem 2 and the details of further applications of these arguments will be given in another paper.

Let us consider, however, the application of Theorems 1 and 2 to the "self-adjoint" case, i.e. when \( A \) and \( B \) are the Euler-Lagrange operators of multiple integral variational problems.

**Theorem 3.** Let \( T \) and \( S \) be the derivatives of two \( C_1 \) functions \( f \) and \( g \) on \( V \), respectively, where \( T \) is bounded and satisfies condition \((S) \) and \( S \) is compact. Let \( c \) be a constant such that on the level set \( M_c = \{ u | f(u) = c \} \), \( (T(u), u) > 0 \), while \( M_c \) is bounded. Suppose that \( g(u) > 0 \) for \( u \) in \( M_c \), that \( (S(u), u) > 0 \) on \( M_c \), and that for each set \( B \) on \( M_c \) for which \( g(u) > \epsilon > 0 \), \( (S(u), u) > d(\epsilon) > 0 \).

Then \( g \) assumes its maximum at a point \( u_0 \) of \( M_c \), and \( T(u_0) = \lambda S(u_0) \) for some \( \lambda > 0 \).

**Proof of Theorem 3.** \( V \) is assumed as in Theorem 1 to be a separable reflexive Banach space. We choose an increasing sequence \( V_n \) of finite dimensional subspaces whose union is dense in \( V \) and with \( M_c \cap V_n \) having their union dense in \( M_c \). Let \( f_n \) and \( g_n \) be the restrictions of \( f \) and \( g \) to \( V_n \). Then \( M_c \cap V_n \) is the \( c \)-level set of \( f_n \) and \( f_n' = j_n^* T, g_n' = j_n^* S \). Since \( (f_n'(u), u) = (T(u), u) > 0 \) on \( M_c \cap V_n, M_c \cap V_n \) is a manifold. The function \( g \) is \( C^2 \) on this compact manifold and assumes its maximum \( m_n \) on \( M_c \cap V_n \) at a point \( u_n \) which satisfies the condition \( T(u_n) = \lambda_n S(u_n) \). Since \( g(u_n) = m_n \rightarrow m = \sup_{u \in M_c} g(u) \), \( (S(u_n), u_n) \geq d_0 > 0 \) for all \( n \). Hence, since

\[ \lambda_n = \frac{(T(u_n), u_n)}{(S(u_n), u_n)}, \]
\(\lambda_n\) is uniformly bounded. If we apply Theorem 1, we obtain the conclusion that for an infinite subsequence, \(u_{n(k)} \to u\), where \(u\) is an eigenfunction \(T(u) = \lambda S(u)\). Since \(g\) is continuous, \(g(u_{n(k)}) \to g(u) = m\). Since \(M_\varepsilon\) is closed, \(u \in M_\varepsilon\).

**THEOREM 4.** Let \(V\) be a separable reflexive Banach space, \(T\) and \(S\) two continuous mappings of \(V\) into \(V^*\) with \(T\) bounded and satisfying condition (\(S\))\(_0\), \(S\) compact. Suppose that \(T\) and \(S\) are the derivatives of two \(C^1\) functions \(f\) and \(g\) on \(V\), and suppose that on the level set \(M_\varepsilon = \{u | f(u) = c\}\), \((T(u), u) > 0\). Suppose that \(M_\varepsilon\) is invariant under the involution \(\pi(u) = -u\), and that \(g(-u) = g(u)\) on \(M_\varepsilon\). Suppose further that \(M_\varepsilon\) is intersected exactly once by each ray through the origin, that \(g(u) > 0\) for \(u \in M_\varepsilon\), that \((S(u), u) > 0\) on \(M_\varepsilon\), and that \(g(u)\) and \((S(u), u)\) go to zero together on \(M_\varepsilon\). Suppose finally that for each \(\varepsilon > 0\), there exists a finite dimensional subspace \(V_\varepsilon\) of \(V\) such that outside the \(\varepsilon\)-neighborhood of \(V_\varepsilon\), \(g(u) < \varepsilon\). For each \(j\), let

\[
h_j = \sup_{\text{cat } (K, M_\varepsilon) \leq j} \min_{u \in K} g(u),
\]

where the supremum is taken over compact subsets \(K\) of \(M_\varepsilon\) whose image in \(M_\varepsilon/\pi\) has Lusternik-Schnirelman category \(\geq j\).

Then:

(a) For each \(j\), \(h_j\) is well defined and there exists \(u_j\) in \(M_\varepsilon\) with

\[
T(u_j) = \lambda_j S(u_j), \quad (\lambda_j > 0), \quad f(u_j) = c, \quad g(u_j) = h_j,
\]

while \(\lambda_j \to +\infty\), \(h_j \to 0\).

(b) Suppose that \(\dim(V_\varepsilon) \geq j\). Then we can define

\[
h_{j,n} = \sup_{\text{cat } (K, M_\varepsilon) \leq j, K \subset V_\varepsilon} \min_{u \in K} g(u),
\]

and for each \(j \leq n\), there exists \(u_{j,n}\) in \(V_\varepsilon\) such that

\[
f_{n}^* T(u_{j,n}) = f_{n}^* S(u_{j,n}), \quad f(u_{j,n}) = c, \quad g(u_{j,n}) = h_{j,n}.
\]

(c) For any fixed \(j\) and any infinite subsequence \(u_{j,n(k)} \to u_j\) as \(k \to \infty\), \(u_j\) is an eigenfunction satisfying the condition of part (a) and \(u_{j,n(k)} \to u_j\).

**PROOF OF THEOREM 4.** Since \(f_{n}' = f_{n}^* T\), so that \((f_{n}'(u), u) > 0\) on \(M_\varepsilon \cap V_\varepsilon\), the latter is a manifold for each \(n\), and \((M_\varepsilon \cap V_\varepsilon)/\pi\) is homeomorphic to \(P^{n-1}\), which has Lusternik-Schnirelman category \(n\). The conclusions of (b) then follow from the classical Lusternik-Schnirelman theory on finite dimensional manifolds (Lusternik [7], Vainberg [8]). The conclusion of (a) will follow from that of part (c) so that it suffices to prove (c).
Proof of (c). We may assume without loss of generality that $u_{j,n} \to u_j$ as $n \to \infty$. Since $g(u_{j,n}) = h_{j,n} \to h_j$ as $j \to +\infty$ where $h_j > 0$ for each $j$, it follows that $(S(u_{j,n}), u_{j,n}) \geq d_0 > 0$ for all $n$. Hence $\lambda_{j,n} = (T(u_{j,n}), u_{j,n})(S(u_{j,n}), u_{j,n})^{-1}$ is uniformly bounded. Applying Theorem 1, we find that $u_{j,n} \to u_j$. Hence $f(u_j) = \lim_{n} f(u_{j,n}) = c$. Since $g(u_j) = \lim_n g(u_{j,n}) = h_j$, and since by Theorem 1, $u_j$ is an eigenfunction of the pair $(T, S)$, our conclusion follows. Q.E.D.

Remarks. (1) The result of Theorem 4 combined with Theorem 2 generalizes the writer's results in [4] under weaker regularity and boundedness hypotheses on the $A_\alpha$ and makes no explicit use of the theory of infinite dimensional manifolds.

(2) An earlier attempt to weaken the regularity hypotheses of [4] was made by M. Berger [1] using an infinite dimensional argument. His argument in [1] contains a number of serious errors and gaps which make it doubtful that the argument can be carried through (cf. the review by C. W. Clark in Math. Reviews).

(3) A recent paper with a similar title by S. Hildebrandt [6] has no intersection with the present paper since it concerns linear operators depending nonlinearly on $\lambda$, not nonlinear operators depending linearly on $\lambda$. However, the methods of the present paper can be used to combine Hildebrandt's results with those given here and extend them to nonlinear operators.

Bibliography


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