SEMICONTRACTIVE AND SEMIACCRETIVE NONLINEAR MAPPINGS IN BANACH SPACES

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Let X be a real Banach space, U and T mappings of a subset G of X into X. Then U is said to be nonexpansive if for all u and v in G,

$$(1) \qquad \|U(u) - U(v)\| \leqq \|u - v\|,$$

while T is said to be accretive if for all u and v of G,

$$(2) \qquad (T(u) - T(v), J(u - v)) \geqq 0,$$

where J is a mapping of X into its adjoint space X^* such that for all u in X, $(J(u), u) = \|u\|^2$ and $\|J(u)\| = \|u\|$.

In some recent papers ([7], [8], [9]), we have presented an existence theory for solutions of nonlinear functional equations in uniformly convex Banach spaces X involving nonexpansive and accretive mappings. These results were obtained by interweaving the fixed point theory of nonexpansive mappings with the theory of initial value problems for differential equations in X involving accretive operators. It is our object here to sharpen this theory and to use the sharpened form to extend the preceding results to more general classes of operators obtained by compact perturbation from nonexpansive or accretive operators. When X is a Hilbert space (or, more generally, has a weakly continuous duality mapping), such results were obtained earlier by the writer in [1], [2], [4]. The methods used there involving monotone operators do not apply in our more general context.

We begin by defining two basic classes of nonlinear mappings, the first generalizing the mappings of the form $U+C$ with U nonexpansive and C completely continuous, and the second, the mappings of the form $T+C$ with T accretive and C completely continuous. (We recall that a map C of X into X is said to be completely continuous if it carries weakly convergent sequences in X into strongly convergent sequences in X.)

DEFINITION 1. *Let X be a Banach space, G a subset of X, U a mapping of G into X. Then U is said to be semicontractive if there exists a mapping V of $G \times G$ into X such that $U(u) = V(u, u)$ for u in G, while:*
  (a) *For each fixed v in G, $V(\cdot, v)$ is nonexpansive from G to X.*
  (b) *For each fixed u in G, $V(u, \cdot)$ is completely continuous from G*

to X, uniformly for u in bounded subsets of G, (i.e. if v_j converges weakly to v in G and $\{u_j\}$ is a bounded sequence in G, then

$$V(u_j, v_j) - V(u_j, v) \to 0, \quad \text{strongly in } G).$$

DEFINITION 2. *Let X be a Banach space, G a subset of X, T a map of G into X. Then T is said to be semiaccretive if there exists a mapping S of $G \times X$ into X such that $T(u) = S(u, u)$ for u in G, while:*

(a) *For each fixed v in X, $S(\cdot, v) = S_v$ is an accretive map from G to X such that either $G = X$ and S_v is continuous or $S_v = S_{1,v} + S_{0,v}$ where $S_{1,v} + I$ has all of X as its range and $S_{0,v}$ is the restriction to G of a bounded continuous accretive mapping of X into X.*

(b) *For each fixed u in G, $S(u, \cdot)$ is completely continuous from X to X, uniformly for u in bounded subsets of G.*

Our basic results are the following theorems:

THEOREM 1. *Let X be a uniformly convex Banach space, G a closed bounded convex subset of X with 0 in the interior of G. Let U be a semicontractive mapping of G into X such that for each x in the boundary of G, $U(x) \neq \lambda x$ for any $\lambda > 1$. Then U has a fixed point in G.*

THEOREM 2. *Let T be a semiaccretive mapping of G into X as defined in Definition 2, where X and X^* are uniformly convex. Suppose that $T^{-1}(B)$ is bounded for each bounded set B. Then the range of T is all of X.*

Theorems 1 and 2, as well as other more precise results, follow from the following basic properties of semicontractive and semiaccretive operators given in Theorems 3 and 4, together with the more general existence theorems, Theorems 5 and 6.

THEOREM 3. *Let X be uniformly convex, G a closed bounded convex subset of X, U a semicontractive mapping of G into X. Then:*

(a) *$(I - U)$ is demiclosed, i.e. if u_j converges weakly to u_0 in G and $(I - U)u_j$ converges strongly to w, then $(I - U)(u_0) = w$.*

(b) *$(I - U)(G)$ is closed in X.*

THEOREM 4. *Let X be uniformly convex with X^* uniformly convex, T a semiaccretive mapping from a subset G_0 of X into X (as in Definition 2). Then T is demiclosed and for any closed bounded convex subset G of X, $T(G \cap G_0)$ is closed in X.*

THEOREM 5. *Let X be a Banach space, G a closed bounded convex subset of X having 0 in its interior, U a mapping of G into X such that for each x on the boundary of G, $U(x) \neq \lambda x$ for any $\lambda > 1$. Suppose that*

*for a given constant $k \leq 1$ and a mapping V of $G \times G$ into X, $U(u)$
$= V(u, u)$ for all u in G while*

$$\|V(u, w) - V(v, w)\| \leq k\|u - v\|, \ (u, v, w \in G),$$

*and the map $u \to V(\cdot, u)$ is compact from G to the space of maps from G
to X with the uniform metric.*

Then:

(a) *If $k < 1$, U has a fixed point in G.*

(b) *If $k \leq 1$ and $(I - U)(G)$ is closed in X, then U has a fixed point
in G.*

THEOREM 6. *Let X be a Banach space with X^* uniformly convex, T
a semiaccretive mapping from a subset of X into X (in the sense of
Definition 2). Suppose that $T^{-1}(B)$ is bounded for each bounded ball B
in X, and that for each ball about the origin, $T(B)$ is closed in X.*

Then the range of T is the whole space X.

Theorem 1 follows from the Theorems 3 and 5, while Theorem 2
follows from Theorems 4 and 6.

PROOF OF THEOREM 3. For any demiclosed map T and any weakly
compact subset G, $T(G)$ is closed. Hence it suffices to prove $(I - U)$
demiclosed. If we replace U by $U + w_0$, w_0 fixed, we may take $w = 0$,
and we have $u_j \to u_0$ weakly, $(I - U)u_j \to 0$ strongly, and it suffices to
show that $(I - U)(u_0) = 0$. By condition (b) of Definition 1, $V(u_j, u_j)$
$- V(u_j, u) \to 0$ strongly. Hence $u_j - V(u_j, u) \to 0$ strongly. Replacing
U by $V(\cdot, u)$, we may assume without loss of generality that U is
nonexpansive.

Since X is uniformly convex, for each $\epsilon > 0$, there exists a largest
$\xi(\epsilon) > 0$, $\xi(\epsilon) \leq \epsilon$ such that if for any three elements x, u, and v of X
and any λ with $0 \leq \lambda \leq 1$, we have $\|x - u\| \leq \|u_\lambda - u\| + \xi(\epsilon)$, $\|x - v\|$
$\leq \|u_\lambda - v\| + \xi(\epsilon)$, with $u_\lambda = \lambda u + (1 - \lambda)v$, then

$$\|x - u_\lambda\| \leq \epsilon\|u - v\|.$$

We assume diam $G \leq 1$ and following a device of Göhde [11], for a
given $\epsilon_0 > 0$, we let $\{\epsilon_j\}$ be the sequence defined recursively by
$\epsilon_j = \xi(\epsilon_{j-1})$. We note that if $\|u - U(u)\| \leq \epsilon_j$, $\|v - U(v)\| \leq \epsilon_j$, and let
$u_\lambda = \lambda u + (1 - \lambda)v$ for λ in $[0, 1]$, then $\|U(u_\lambda) - u\| \leq \|U(u_\lambda) - U(u)\|$
$+ \|u - U(u)\| \leq \|u_\lambda - u\| + \epsilon_j$. Similarly, $\|U(u_\lambda) - v\| \leq \|u_\lambda - v\| + \epsilon_j$.
Since $\epsilon_j = \xi(\epsilon_{j-1})$, it follows that $\|u_\lambda - U(u_\lambda)\| \leq \epsilon_{j-1}$.

By thinning out the sequence $\{u_j\}$, we may assume that for each
j, $\|u_j - U(u_j)\| \leq \epsilon_j$. If y is any element of the convex span of the se-
quence $\{u_j\}$, after this thinning out process has been carried through,
we assert that $\|y - U(y)\| \leq \epsilon_0$. Indeed, for y in the convex span of

$[u_j, \cdots, u_k]$ (written with ascending indices), $\|y - U(y)\| \leq \epsilon_{j-1}$. For two elements $[u_j, u_k]$, this follows from the preceding paragraph. If it is true for the s elements $[u_{j_1}, \cdots, u_k]$, it is true for $(s+1)$ elements $[u_j, u_{j_1}, \cdots, u_k]$ since an element of the convex span of the latter set is a convex linear combination of u_j and of an element of the convex span of $[u_{j_1}, \cdots, u_k]$. Since u_0 is the weak limit of the thinned-out sequence $\{u_j\}$, it lies in the convex span of the sequence. Hence $\|u_0 - U(u_0)\| \leq \epsilon_0$. Since $\epsilon_0 > 0$ was arbitrary, it follows that $u_0 = U(u_0)$.
Q.E.D.

PROOF OF THEOREM 4. As in the proof of Theorem 3, it suffices to prove that T is demiclosed, and we may replace T by $S(\cdot, u_0)$ and assume that T is accretive and satisfies condition (a) of Definition 2. By the results of [7], for each v in G_0, the domain of T, the initial value problem

$$(3) \qquad du/dt = -T(u), \qquad t \geq 0; u(0) = v,$$

has one and only one weakly continuously differentiable solution u on R^+, the mapping $U(t)$ for each $t \geq 0$ given by $U(t)v = u(t)$ is a non-expansive map of G_0 into X, and for all $t \geq 0$, $\|T(u(t))\| \leq \|T(v)\|$.

Suppose then that $u_j \to u_0$ weakly and that $T(u_j) \to 0$ strongly in X. If G is the convex closure of the sequence $\{u_j\}$, it suffices to show that there exists an element u' in the closed bounded convex subset G for which $T(u')$ is defined and equals 0. As we have already noted in [8], such an element u' is precisely one lying in the common fixed point set of the mappings $U(t)$, $t > 0$. To show that this common fixed point set is nonvoid, it suffices to show, since the mappings $U(t)$ commute, that each $U(t)$ has a fixed point in G. However,

$$\|u_j - U(t)u_j\| \leq \int_0^t \|T(U(t))\, u_j\| dt \leq t\|T(u_j)\| \to 0 \qquad (j \to +\infty).$$

Applying Theorem 3, we see that $U(t)$ has a fixed point in G. Q.E.D.

PROOF OF THEOREM 5. We note first that conclusion (b) of Theorem 5 follows from conclusion (a). Indeed, if U is a mapping satisfying the conditions of (b), then for each k with $0 < k < 1$, the mapping $U_k(x) = kU(x)$ satisfies the condition of (a) with $V_k(u, v) = kV(u, v)$, and on the boundary of G, $U_k(x) = \lambda x$ implies that $U(x) = \lambda k^{-1}x$. Applying the conclusion of (a), we see that U_k has a fixed point u_k in G. As $k \to 1$, $u_k - U(u_k) = (k-1)U(u_k) \to 0$. If $(I - U)(G)$ is closed, it follows that U has a fixed point in G.

The proof of conclusion (a) follows a variant of the argument of [5]. For each t with $0 \leq t \leq 1$, we set $U_t(x) = tU(x)$. Our hypothesis on the nonexistence of eigenfunctions of U on the boundary of G with

eigenvalues $\lambda > 1$, implies that either U has a fixed point on the boundary of G, (the trivial case), or none of the mappings U_t have fixed points on the boundary of G. For each t and each w in G, the mapping $V_t(\cdot, w)$ is a strict contraction on G with values in X. Hence the image of the interior of G under the mapping $I - V_t(\cdot, w)$ is an open subset $G_{t,w}$ of X and $(I - V_t(\cdot, w))$ maps the boundary of G on the boundary of $G_{t,w}$. For each v in $\overline{G}_{t,w}$, there exists exactly one element u of G such that $u - V_t(u, w) = v$. We denote this element by $C_{t,v}(w)$. Since the map $w \to V(\cdot, w)$ is a compact map of G into the space of maps from G to X with the uniform metric, $C_{t,v}$ is a compact map of G into X for each fixed t and v, and indeed the union of the images over all t in $[0, 1]$ and all v in G of the maps $C_{t,v}$ is relatively compact in X.

For each t in $[0, 1]$, we now let $\psi_t = \{ w \mid w \in G, \ 0 \in \overline{G}_{t,w} \}$ and for each w in ψ_t, we define $Q_t(w) = C_{t,0}(w)$. Similarly, we define ψ as the subset of $X \times I$ consisting of those points $[t, w]$ such that w lies in ψ_t, and we set $Q(t, w) = Q_t(w)$. We note that if w lies in the boundary of ψ_t, then $C_{t,0}(w)$ is well defined and lies on the boundary of G. Hence Q may be extended by continuity to a compact mapping of the closure of ψ into X, and maps all points of the boundary of ψ into the boundary of G. Moreover, ψ is an open subset of $X \times I$, since if 0 lies in the image of $I - V_t(\cdot, w)$ acting on the interior of G for a given pair $[t, w]$, then for $[s, z]$ near $[t, w]$, the equation $u - V_s(u, z) = 0$ must have a solution u in the interior of G since the image of the interior of G by the homeomorphism $I - V_s(\cdot, z)$ must contain an entire component of $X - (I - V_s(\cdot, z))(\text{bdry } G)$. By choosing $[s, z]$ near $[t, w]$, we can ensure that the boundary image above does not intersect a given ball about 0 while the image of $C_{t,0}(w)$ under $(I - V_s(\cdot, z))$ lies in that ball. Hence the whole ball including the point 0 is covered by the image of $(I - V_s(\cdot, z))$.

To complete the proof, we apply the Leray-Schauder theory [14]. For $t = 0$, Q_0 is the constant map 0 defined on $\psi_0 = G$, and the degree of $(I - Q_0)$ on ψ_0 must equal $+1$. For all t and all x on the boundary of ψ_t, $Q_t x$ is never equal to x, since if it were, we should have $x - V_t(x, x) = x - U_t(x) = 0$ with $x = Q_t x$ lying on the boundary of G. Hence Q_1 has a fixed point in ψ_1, which implies that $x - U(x) = 0$ has a solution in G.   Q.E.D.

The proof of Theorem 6, which is another variant of the argument of [5], will be given in detail elsewhere.

In conclusion, we note that the result of Theorem 3 yields the following generalization of the result of Browder-Petryshyn [10]:

THEOREM 7. *Let X be a uniformly convex Banach space, G a closed bounded convex subset of X, U a nonexpansive mapping of G into G. For a constant t with $0 < t < 1$, let $U_t = tU + (1 - t)I$. Suppose that U has at most one fixed point y in G. Then for each x_0 in G, the sequence $(U_t)^n x_0$ converges weakly to y in X.*

PROOF OF THEOREM 7. Let $x_n = (U_t)^n x_0$ for a given x_0 and $n \geq 1$. To show that x_n converges weakly to y, it suffices to show that each weakly convergent subsequence of $\{x_n\}$ converges to a fixed point of U. It suffices by the demiclosedness of U to show that $(I - U)(x_n) \to 0$. Since U_t is also nonexpansive and y is a fixed point of U_t, $\|x_n - y\|$ is nonincreasing in n, and hence converges. On the other hand, $x_{n+1} - y = t(U(x_n) - U(y)) + (1 - t)(x_n - y)$, where $\|U(x_n) - U(y)\| \leq \|x_n - y\|$. By the uniform convexity of X, it follows that $U(x_n) - x_n \to 0$.   Q.E.D.

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