A UNIFORM GENERALIZED SCHOENFLIES THEOREM

BY PERRIN WRIGHT

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The generalized Schoenflies theorem of M. Brown [2], [3] can be restated in the following way: If $S^{n-1}$ is the equator of $S^n$, then any locally flat embedding $f: S^{n-1} \to S^n$ can be extended to a homeomorphism $F: S^n \to S^n$.

The purpose of this paper is to show that, if $n \geq 5$, the extension $F$ can be constructed in a controlled manner; in particular, if $f: S^{n-1} \to S^n$ is close to the inclusion embedding, then $F: S^n \to S^n$ can be chosen to be close to the identity homeomorphism. Consequently if, $f, g : S^{n-1} \to S^n$ are locally flat embeddings, $n \geq 5$, and $f$ is close to $g$, then there is a homeomorphism $H: S^n \to S^n$ which is close to the identity such that $Hf = g$.

Let $S^n$ denote the unit sphere in $E^n$, $B^n$ the unit ball, and $O$ the origin. If $x, y$ belong to $E^n - O$, let $\theta(x, y)$ denote the angle in radians between the line segments $Ox$ and $Oy$, measured such that $0 \leq \theta(x, y) \leq \pi$. The distance between $x$ and $y$ under the Euclidean metric will be denoted by $\text{dist}(x, y)$. If $A$ is a subset of $E^n - O$, the angular diameter of $A$, written $\theta \text{ diam } A$, is defined to be $\sup_{x, y \in A} \theta(x, y)$. This is significant whenever $A$ lies in a half-space.

Now suppose $S$ is a locally flatly embedded $(n-1)$-sphere in $E^n$ which approximates the standard sphere $S^{n-1}$. Suppose $\phi: S^{n-1} \times [0, 1] \to \text{Cl}(\text{Ext } S)$ is a collar on $S$ in $\text{Cl}(\text{Ext } S)$. If the collar is small, then the $\theta$-diameter of each fiber $\phi(x \times [0, 1])$ is also small. The object of Lemma 2 is to push the collar outward, leaving $S$ fixed, so that its two boundary components are separated by a round sphere with center at $O$, and so that the $\theta$-diameter of each fiber remains small. The precise statement is as follows.

**Lemma 2.** If $f: S^{n-1} \to E^n$, $n \geq 5$, is a locally flat embedding such that for all $x \in S^{n-1}$, $\theta(x, f(x)) < \epsilon$, where $\epsilon < \pi/7$, then there is an embedding $F: S^{n-1} \times [0, 1] \to \text{Cl}(\text{Ext } f(S^{n-1}))$ such that:

1. $F(x, 0) = f(x)$,
2. $F(S^{n-1} \times 0)$ and $F(S^{n-1} \times 1)$ are separated by some round sphere with center at $O$,
3. For all $x \in S^{n-1}, t \in [0, 1]$, $\theta(x, F(x, t)) < 13\epsilon/2 + 15\epsilon$.

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The proof of Lemma 2 requires four auxiliary lemmas. We begin with a collar $\phi$ on $S = \phi(S^{n-1})$, and let $U = \phi(S^{n-1} \times (0, 1))$. Then $U$ is an open subset of $\text{Ext} S$. We further assume $\theta(x, \phi(x, t)) < \epsilon$ for all $x$ and $t$. Lemma A states that any complex in $\text{Ext} S$ can be pulled into $U$ (in the sense of [1]) by a homotopy in $\text{Ext} S$ whose orbits have $\theta$-diameter at most $9\epsilon$. Lemma B states that any complex in $\text{Ext} S$ can be disentangled from $S$, i.e., pulled into the exterior of some round sphere $\Sigma$ outside $S$, by a homotopy in $\text{Ext} S$ whose orbits have $\theta$-diameter less than $4\epsilon$. The condition that $\theta(x, \phi(x)) < \epsilon$ for all $x \in S^{n-1}$ insures that the "folds" in $S$ are small, and hence any point of $\text{Ext} S$ may be moved into $U$ or outside $\Sigma$ along a path of small $\theta$-diameter. The condition $\epsilon < \pi/7$ is a purely artificial one which makes the proofs work.

Lemmas 1A and 1B are radial engulfing lemmas. The engulfings proceed along the orbits of the homotopies guaranteed by Lemmas A and B. The proofs of these lemmas are almost identical to the proof of Engulfing Theorem A of [1], and their functions are comparable to those of Lemmas 1 and 2 of [5].

Finally, the proof of Lemma 2 is accomplished in the manner of Lemma 9.1 of [6].

If $S_1$ and $S_2$ are disjoint locally flat $(n-1)$-spheres in $E^n$, $S_1 \subset \text{Int} S_2$, and if there is a stable homeomorphism $h: E^n \to E^n$ such that $h(S_1) = S_2$, then $S_1$ and $S_2$ cobound an annulus (Theorem 10.3 of [4]). We next strengthen a special case of this theorem.

Let $\bar{S}$ be a sphere concentric with $S^{n-1}$. Let $x$ denote the point of $\bar{S}$ which is coradial with $x \in S^{n-1}$. Introduce the following notation: if $y \in E^n - 0$ and $L$ is a real number such that $||y||+L > 0$, then $y+L$ denotes the unique point of $E^n$ which is coradial with $y$ and has norm $||y||+L$.

If $f: S^{n-1} \to E^n$ and $\bar{f}: \bar{S} \to E^n$ are embeddings, we say that $f$ and $\bar{f}$ are parallel if there is a real number $L$ such that for all $x \in S^{n-1}$, $f(x) = \bar{f}(x) + L$.

Clearly any two disjoint parallel spheres are stably equivalent, hence cobound an annulus. Lemma 3 states that this annulus can be coordinatized so that the $\theta$-diameters of the fibers are directly proportional to the $\theta$-deviation of $f$ itself.

**Lemma 3.** Let $\bar{S}$ be a sphere concentric with $S^{n-1}$, of radius less than 1. Let $A$ be the annulus between $\bar{S}$ and $S^{n-1}$. Let $0 < \epsilon < \pi/7$, and let $f: S^{n-1} \to E^n$, $n \geq 5$, be a locally flat embedding such that $\theta(x, f(x)) < \epsilon$ for all $x \in S^{n-1}$. Suppose $\bar{f}: \bar{S} \to \text{Int} f(S^{n-1})$ is an embedding which is parallel to $f$. Then there is an embedding $F: A \to E^n$ such that:
(1) \( F|_{S^{n-1}} = f \),
(2) \( F|_S = \tilde{f} \),
(3) \( \theta(y, F(y)) < (39/2)ne + 45\varepsilon \), for all \( y \in A \).

To prove Lemma 3, apply Lemma 2 to obtain an annulus in \( \text{Cl}(\text{Ext } f(S^{n-1})) \) which satisfies the conclusion of Lemma 2. Call this annulus \( R_1 \) and denote the annulus between \( f(S^{n-1}) \) and \( \tilde{f}(S) \) by \( R_2 \). Using the fact that \( \text{Int } R_1 \) contains a round sphere with center at \( O \), push \( R_1 \) onto \( R_1 \cup R_2 \) by a radial homeomorphism of \( E^n \). This does not alter the \( \theta \)-diameters of the fibers of \( R_1 \). Next, map \( R_1 \cup R_2 \) homeomorphically onto \( R_1 \) by utilizing the annular structure on \( R_1 \). This at worst triples the \( \theta \)-diameters of fibers. The result of these maps gives an annular structure on \( R_2 \) satisfying Lemma 3.

The main theorems.

**Theorem 1.** If \( n \geq 5 \), and \( f: S^{n-1} \to E^n \) is a locally flat embedding such that \( \theta(f(x), x) < \varepsilon \) and \( \text{dist}(f(x), x) < \varepsilon \) for all \( x \in S^{n-1} \), then \( f \) can be extended to an embedding \( F: B^n \to E^n \) such that \( \text{dist}(F(x), x) < 39ne/2 + 48\varepsilon \).

**Corollary 1.** For each \( \eta > 0 \), there is a \( \delta > 0 \) such that each locally flat \( \delta \)-embedding of \( S^{n-1} \) into \( E^n \), \( n \geq 5 \), can be extended to an \( \eta \)-embedding of \( B^n \) into \( E^n \).

The proof of Theorem 1 is outlined as follows. Partition \( B^n \) into annuli \( A_i \) of thickness \( 2\varepsilon \) together with a small ball \( B_* \) in the center. Partition \( \text{Cl}(\text{Int } f(S^{n-1})) \) into annular regions \( R_i \) together with a small cell \( C \) about the origin, in such a way that each boundary sphere of each \( R_i \) is parallel to \( f(S^{n-1}) \) and the parallel distance between any two consecutive spheres (i.e., the constant \( L \) of the definition of parallel embeddings) is \( 2\varepsilon \). Obtain a 1-1 correspondence between the \( A_i \) and the \( R_i \) by omitting the innermost \( A_i \) or \( R_i \), if necessary. Use Lemma 3 to map the outermost annulus \( A_0 \) homeomorphically onto the outermost region \( R_0 \). (We assume \( \varepsilon < \pi/7 \), for if not, Theorem 1 is certainly true.) Then it is possible to map each \( A_i \) onto the corresponding \( R_i \) by copying the map \( f|_{A_0} \). This procedure is well defined on \( A_i \cap A_{i+1} \), because of the parallel condition. Finally, map \( B_* \) homeomorphically onto \( C \) in any fashion, extending the map \( F|_{B_*} \).

For points \( y \in A_i \), \( \|y\| - \|F(y)\| < 3\varepsilon \) and \( \theta(y, F(y)) < (39/2)ne + 45\varepsilon \). Since \( \|y\| \leq 1 \), \( \text{dist}(y, F(y)) < (39/2)ne + 48\varepsilon \). For points \( y \in B_* \), no control is necessary because \( B_* \cup C \) has diameter less than \( 7\varepsilon \).

Now consider \( S^{n-1} \) to be the equator of \( S^n \). Theorems 2 and 3 follow from Corollary 1.
Theorem 2. Let \( n \geq 5, \eta > 0 \). There is a \( \delta > 0 \) such that any locally flat \( \delta \)-embedding \( f: S^{n-1} \rightarrow S^n \) can be extended to a \( \eta \)-homeomorphism \( F: S^n \rightarrow S^n \).

Theorem 3. Let \( n \geq 5, \eta \geq 0 \). Let \( g: S^{n-1} \rightarrow S^n \) be any locally flat embedding. There exists a \( \delta > 0 \) such that if \( f: S^{n-1} \rightarrow S^n \) is any locally flat embedding satisfying \( \text{dist}(f(x), g(x)) < \delta \), then there is an \( \eta \)-homeomorphism \( H: S^n \rightarrow S^n \) such that \( Hf = g \).

These results, together with those of Connell [5] and Bing [1], can be used to show that the problem of approximating homeomorphisms of \( S^n, n \geq 5 \), by p.w.l. ones is equivalent to approximating locally flat embeddings of \((n-1)\)-spheres by p.w.l. ones.

References


Florida State University