THE COHOMOLOGICAL DIMENSION OF STONE SPACES

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The purpose of this note is to announce a few inequalities involving
the cohomological (sheaf-theoretic) dimension of locally compact,
totally disconnected Hausdorff spaces, herein called Stone spaces.
Throughout, $R$ will denote a commutative regular ring with maximal
ideal space $X$. (Then $X$ is compact and totally disconnected.) For
each ideal $J$ in $R$ let $U[J]$ denote the corresponding open subset of $X$,
and for each $R$-module $A$, let $\mathcal{G}(A)$ denote the corresponding sheaf of
modules, as defined in [2].

Theorem 1. $\text{Ext}^n_R(J, A)$ and $H^n(U[J]; \mathcal{G}(A))$ are naturally iso­

morphic.

Theorem 2. Let $\mathcal{F}$ be a sheaf over the Stone space $X$, and let $\mathcal{U}$ be a
covering of $X$ consisting of compact open sets. Then the natural maps
$H^n(\mathcal{U}; \mathcal{F}) \to H^n(X; \mathcal{F}) \to H^n(X; \mathcal{F})$ are all isomorphisms.

Let $\dim X$ denote the cohomological dimension of $X$, and $\text{cov dim } X$
the covering dimension of $X$, based on arbitrary (not necessarily
finite) open coverings. (It is not hard to show that for Stone spaces,
$\text{cov dim } X \leq n$ iff $X$ has a compact open cover of order $n$.) Finally,
let $\text{h-dim}_R J$ denote the homological (projective) dimension of
the ideal $J$.

Corollary. $\text{h-dim}_R J \leq \dim U[J] \leq \text{cov dim } U[J].$

Since the only projective $R$-modules are direct sums of principal
ideals [1], we see that $\text{h-dim}_R J = 0$ iff $\text{cov dim } U[J] = 0$, and, by
the corollary, iff $\dim U[J] = 0$. In order to see that equality need not
always hold in the corollary, let us define the rank $\rho$ of a space $X$ by
agreeing that $\rho(X) \leq n$ iff $X$ can be written as a union of $\mathbb{N}_n$ (or fewer)
compact sets.

Theorem 3. For any Stone space $X$, $\dim X \leq \rho(X)$.

Example 1. Let $\Omega$ be the set of countable ordinals, with the order
topology. Then $\dim \Omega = 1$, but $\text{cov dim } \Omega = \infty$. (The second assertion
may be verified directly; the first then follows from Theorem 3 and
the remarks following the corollary.)

The next example shows that the inequality in Theorem 3 cannot
be sharpened.
EXAMPLE 2. For each $n \geq 0$, let $X_n$ be the product of $n$ copies of a two-point space, with a single point deleted. Then $\dim X_n = \rho(X_n) = n$. (Pierce [3] has shown that the corresponding maximal ideal in the free Boolean ring on $\aleph_n$ generators has homological dimension $n$. Therefore $\dim X_n = n$, by Theorem 3 and the corollary.)

EXAMPLE 3. Let $A_0$ (resp. $A_1$) be the one-point compactification of a discrete space of cardinality $\aleph_0$ (resp. $\aleph_1$). Let $X = A_0 \times A_1 \setminus \{(*, *)\}$. Then $\dim X = \text{cov dim } X = \rho(X) = 1$. (In fact, it can be shown that $H^1(X; \mathbb{Z}_2) \neq 0$, where $\mathbb{Z}_2$ denotes the constant 2-sheaf.)

I do not know whether the identity $\dim X = \text{cov dim } X = n$ can be realized in general. (The space $X_n$ of Example 2 has infinite covering dimension.) An obvious generalization of Example 3 yields a space with rank and covering dimension $n$, but with unknown cohomological dimension. Also, I know of no example in which $h \cdot \text{dim}_R J < \dim U[J]$. Notice that if one could show that $h \cdot \text{dim}_R J$ and $\dim U[J]$ are always equal, then it would follow that any two commutative regular rings with homeomorphic maximal ideal spaces have the same global dimension.

REFERENCES


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