THE STRUCTURE OF TORSION ABELIAN GROUPS GIVEN BY PRESENTATIONS[1]

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Let F_X denote the free abelian group freely generated by the set X, and let R be a subset of F_X. With $[R]$ denoting the subgroup of F_X generated by R, set

$$G(X, R) = F_X/[R],$$

i.e., $G(X, R)$ is that abelian group generated by X and subject only to the relations

$$r = 0 \qquad \text{all } r \in R.$$

If each of the elements in R involves only one generator in X, then $G(X, R)$ is a direct sum of cyclic groups. On the other hand, if G is any abelian group, then $G \cong G(X, R)$, where each element in R involves at most three generators in X; indeed this isomorphism results if we take $X = G$ and R equal to the set of all elements in F_G of the form $x+y-z$, where $z = x+y$ in G.

Our purpose here is to investigate the structure of the group $G(X, R)$ in the intermediate case when each of the elements of R involves at most two generators, and $G(X, R)$ is a torsion group. We can evidently restrict our attention to p-groups, and in this case it is easily seen that $G(X, R) \cong G(X', R')$, where each element in R' is of one of the forms

$$p^n x \quad \text{or} \quad p^n x - y.$$

This leads us to the following definition. Let X be a set, V be a subset of the set of ordered pairs $\langle x, y\rangle$ with $x, y \in X$, u be a map of X to the nonnegative integers, and v be a map of V to the nonnegative integers. By $G(X, V, u, v)$ we mean that abelian group generated by X and subject only to the relations

$$p^{u(x)}x = 0 \qquad \text{all } x \in X,$$
$$p^{v(x,y)}x = y \qquad \text{all } \langle x, y\rangle \in V.$$

We say that an abelian p-group G is a *T-group* if $G \cong G(X, V, u, v)$ for some $\langle X, V, u, v\rangle$.

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One property of T-groups is clear: *the direct sum of a family of T-groups is again a T-group.* Every divisible p-group is certainly a T-group, and the reduced part of a T-group is again a T-group.

Before stating our main results concerning these groups, let us recall a few basic definitions. Let G be any reduced abelian p-group. Define the subgroups $p^\alpha G$ for each ordinal α as usual by the rules: $p^0G=G$; $p^\alpha G=\{px \mid x\in p^{\alpha-1}G\}$ if $\alpha-1$ exists; $p^\alpha G=\bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. Since G is reduced, there is a first ordinal λ, called the *length* of G, such that $p^\lambda G=0$. For each ordinal α we set

$$f_G(\alpha) = \text{rank } p^\alpha G \cap G[p]/p^{\alpha+1}G \cap G[p],$$

where $G[p]=\{x\in G \mid px=0\}$, and we call the cardinal number $f_G(\alpha)$ the αth *Ulm invariant* of G. Finally we let ω denote the first infinite ordinal and Ω denote the first uncountable ordinal.

The description of T-groups is now accomplished by the following theorems.

(A) *If G and H are reduced T-groups, then G and H are isomorphic if and only if $f_G(\alpha)=f_H(\alpha)$ for each ordinal α.*

(B) *Let f be a map of an ordinal λ to a set of cardinal numbers. Then there exists a reduced T-group G of length λ such that $f_G(\alpha)=f(\alpha)$ for each $\alpha<\lambda$, if and only if f satisfies the following conditions:*

(i) $\lambda=\sup\{\alpha+1 \mid f(\alpha)\neq 0\}$;

(ii) *if α is a limit ordinal such that $\alpha+\omega<\lambda$, and $0\leqq\eta<\omega$, then*

$$\sum_{\alpha+\eta\leq\beta<\alpha+\omega} f(\beta) \geqq \sum_{\alpha+\omega\leq\beta<\lambda} f(\beta).$$

(C) *A reduced p-group G is a direct sum of countable groups if and only if G is a T-group of length at most Ω.*

When specialized to countable p-groups, (A) and (C), of course, reduce to Ulm's Theorem, and in the case of direct sums of countable groups they reduce to the theorem of Kolettis [2]. Our results are not independent of Ulm's Theorem, however, since it is used to establish (C). The proofs of (A), (B) and (C) will appear elsewhere.

Actually T-groups have been studied before in a different guise. In [3], Nunke defines a reduced p-group G to be *totally projective* if

$$p^\alpha \text{Ext}(G/p^\alpha G, A) = 0$$

for all ordinals α and every group A, and he obtains a number of properties of these groups. Quite recently Hill [1] has announced that two totally projective groups with the same Ulm invariants are isomorphic. Now it is easily verified that *if G is a reduced T-group and α is* *an ordinal, then both $p^\alpha G$ and $G/p^\alpha G$ are T-groups.* Moreover, (A) and (B) yield that *a T-group whose length is a limit ordinal is a direct sum of T-groups of smaller length.* These last two facts, in conjunction with [3, 2.6], imply that every reduced T-group is totally projective. On the other hand, if H is a totally projective group, then the function f_H necessarily satisfies condition (ii) of (B). Consequently there is a reduced T-group G having the same Ulm invariants as H, and Hill's theorem guarantees that G and H are isomorphic. Thus *a reduced abelian p-group is totally projective if and only if it is a T-group.*

The foregoing results further provide a characterization of the class of all reduced T-groups in terms of certain natural group-theoretic properties. *Let $\mathcal{K}$ be a class of reduced abelian p-groups. Then $\mathcal{K}$ coincides with the class of all reduced T-groups if and only if $\mathcal{K}$ has the following properties*: (1) *$\mathcal{K}$ is closed under isomorphism*; (2) *$\mathcal{K}$ is closed under direct sums*; (3) *if $G \in \mathcal{K}$ and the length of G is a limit ordinal, then G is a direct sum of groups in $\mathcal{K}$ of smaller length*; (4) *for each p-group G and each ordinal α, $G \in \mathcal{K}$ if and only if both $G/p^\alpha G$, $p^\alpha G \in \mathcal{K}$*; (5) *if an abelian p-group G has no elements of infinite height, i.e., $p^\omega G = 0$, then $G \in \mathcal{K}$ if and only if G is a direct sum of cyclic groups.* Thus the class of all reduced T-groups is the smallest class of reduced p-groups that has properties (1)–(4) and contains the finite groups.

REFERENCES

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