THE IMPOSSIBILITY OF DESUSPENDING COLLAPSES

BY W. B. R. LICKORISH AND J. M. MARTIN

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It is known that in order to prove the polyhedral Schoenflies conjecture in all dimensions, it is enough to show that, if \((B^5, B^3)\) is a \((4, 3)\) ball pair, then \(B^4\) collapses (polyhedrally) to \(B^3\). Recently, using the solution to the polyhedral Poincaré conjecture in high dimensions, Husch has shown [3] that if \((B^7, B^6)\) is a \((7, 6)\) ball pair, then \(B^6\) collapses to \(B^6\). It is tempting to try to prove that \(B^6\) collapses to \(B^3\) by invoking the following conjecture.

**Conjecture A.** If \(M\) is a polyhedral manifold, \(L\) a submanifold of \(M\) and \(S(M)\setminus S(L)\), then \(M\setminus L\). (\(S(X)\) denotes the suspension of \(X\) and \("\setminus"\) denotes a polyhedral collapse.)

If Conjecture A were true we could suspend a \((4, 3)\) ball pair three times to obtain a \((7, 6)\) ball pair, use Husch’s result, and then apply Conjecture A three times in order to desuspend the collapse.

In this note we present a counterexample to Conjecture A, and discuss other conjectures related to the problem of desuspending collapses.

**Example 1.** Let \(M^4\) be a polyhedral 4-manifold, as described in [4] or [5], with the following properties, (a) \(M^4\) is contractible, (b) \(\pi_1(\partial M)\neq 0\), (c) \(M^4\times I \cong B^5\). Consider \(S(M^4)\) as \(M^4\times I\) together with a cone on \(M^4\times \{0\}\) and another cone on \(M^4\times \{1\}\). Thus if \(v_0\) and \(v_1\) are the vertices of these cones,

\[
S(M^4) = (M^4 \times I) \cup (v_0 * (M^4 \times \{0\})) \cup (v_1 * (M^4 \times \{1\})).
\]

Now let \(B^3\) be a 3-ball in \(\partial M^4\). Since \(M^4\times I\) is a 5-ball, with \(B^3\times I\) as a face, there is an elementary collapse

\[
M^4 \times I \setminus (M^4 \times \{0\}) \cup (M^4 \times \{1\}) \cup (\partial M^4 - \text{int} B^3) \times I.
\]

Thus there is a collapse

\[
S(M^4) \setminus (v_0 * (M^4 \times \{0\})) \cup (v_1 * (M^4 \times \{1\})) \cup (\partial M^4 - \text{int} B^3) \times I.
\]

Now, by collapsing conewise \(v_i * (M^4 \times \{i\})\) to \(v_i * ((\partial M^4 - \text{int} B^3) \times \{i\})\), for \(i = 0\) and 1, we have \(S(M^4) \setminus S(\partial M^4 - \text{int} B^3)\). However, since \(\pi_1(M^4) = 0\) and \(\pi_1(\partial M^4 - \text{int} B^3) \neq 0\), \(M^4\times \partial M^4 - \text{int} B^3\). This provides a counter-example to Conjecture A.

**Remark 1.** By taking two copies of the above manifold, \(M_1\) and

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$M$, 3-balls $B_1$ and $B_2$ in their boundaries and identifying $\partial M_1 - \text{int } B_1$ with $\partial M_2 - \text{int } B_2$, one can obtain a similar counter-example in which $M = M_1 \cup M_2$ is a 4-ball, and $L = \partial M_1 - \text{int } B_1$ is properly imbedded in $M$.

**Remark 2.** By adapting Example 1, the following can be proved. There exists a polyhedron $X$ and a point $x \in X$ such that $S(X) \setminus S(x)$ but $X$ is not collapsible. Take $X$ as $X = M^4 \cup (x \ast (\partial M^4 - \text{int } B^3))$; i.e. $X$ is the 4-manifold mentioned in Example 1 together with a cone on its boundary less the interior of a 3-ball. Now $S(X) \setminus S(x)$ by a similar argument to that used above. Suppose $X$ is collapsible. Then $X \setminus x$ (as a collapsible polyhedron collapses to any given point). Then $X$ is P.L. homeomorphic to a regular neighbourhood of $x$ in $X$ (regular neighbourhoods in polyhedra are defined and extensively discussed in [2]). $x \ast (\partial M^4 - \text{int } B^3)$ is such a regular neighbourhood, so by the regular neighbourhood uniqueness theorem [2] there is a P.L. homeomorphism

$$h: X, x \rightarrow x \ast (\partial M^4 - \text{int } B^3), x.$$  

Hence restricting $h$ to the points of $X$ which do not have neighbourhoods which are open 4-cells, $h$ maps the 3-sphere $B^3 \cup (x \ast \partial B^3)$ homeomorphically onto $(\partial M^4 - \text{int } B^3) \cup (x \ast \partial B^3)$ which is homeomorphic to $\partial M^4$. This is impossible as $\pi_1(\partial M^4) \neq 0$, and hence $X$, is not collapsible.

We now turn our attention to a problem involving simplicial collapsing. Bing [1] has given an example of a triangulation of a 3-cell which is not collapsible. One would hope to be able to suspend this triangulation to obtain noncollapsible triangulations of the $n$-cell. This leads to Conjecture B.

*Conjecture B.** If $K$ is a complex, $L$ is a subcomplex of $K$, and $S(K) \setminus S(L)$, then $K \setminus L$. ("\setminus" denotes simplicial collapsing.)

We do not know the answer to Conjecture B, but it seems likely that it is false (although it is not difficult to prove it true if $K$ is only two-dimensional). The following question is related to Conjecture B.

**Question 1.** Is there a complex $K$, with subcomplexes $X$ and $Y$ such that $K \setminus X$, $K \setminus Y$, $K \setminus X \cup Y$, but $K \setminus X \setminus Y$?

An affirmative answer to Question 1 would provide a counterexample to Conjecture B as follows: Suppose that $K$, $X$ and $Y$ have the properties stated in Question 1. Consider $S(K)$ as $(a \cup b) \ast K$ where $a$ and $b$ are two points. Now since $K \setminus X$, $S(K) \setminus (a \ast X) \cup (b \ast K)$. Since $K \setminus Y$, $(a \ast X) \cup (b \ast K) \setminus (a \ast X) \cup K \cup (b \ast Y)$. This latter complex collapses simplicially to $(a \ast X) \cup (b \ast Y)$ because $K \setminus (X \cup Y)$. By collapsing conewise towards $a$ and $b$, 

$$(a \ast X) \cup (b \ast Y) \setminus (a \ast (X \cup Y)) \cup (b \ast (X \cup Y)) = S(X \cup Y).$$
Thus

\[ S(K) \subseteq S(X \cap Y) \quad \text{but} \quad K \not\subseteq X \cap Y. \]

Using the manifold \( M^4 \) employed in Example 1, it is possible to show as follows that Question 1 would have an affirmative answer if polyhedral collapsing replaced simplicial collapsing.

**Example 2.** Let \( M^4 \) be the manifold used in Example 1, and let \( B^3 \) be a 3-ball in \( \partial M^4 \) as before. Let \( X \) and \( Y \) be sub-polyhedra of \( M^4 \times I \) defined by

\[
X = (M^4 \times \{0\}) \cup ((\partial M^4 - \text{int} B^3) \times I)
\]
\[
Y = (M^4 \times \{1\}) \cup ((\partial M^4 - \text{int} B^3) \times I).
\]

Using the product structure of \( M^4 \times I \),

\[
M^4 \times I \setminus X \quad \text{and} \quad M^4 \times I \setminus Y.
\]

Because \( M^4 \times I \) is a 5-ball, \( M^4 \times I \setminus X \cup Y \), but \( M^4 \times I \setminus X \cap Y \) since \( X \cap Y = ((\partial M^4 - \text{int} B^3) \times I) \) is not simply connected.

**Question 2.** With \( M^4 \), \( X \) and \( Y \) as in Example 2, is there a triangulation of \( M^4 \times I \), triangulating \( X \) and \( Y \) as subcomplexes, so that \( M^4 \times I \setminus X \), \( M^4 \times I \setminus Y \), and \( M^4 \times I \setminus X \cup Y \)?

*Added in proof.* The answer to Question 2 is "Yes." L. C. Glaser has pointed out that this follows at once from Theorem 7 of J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math Soc. **45** (1939), 243–327. Thus Question 1 has an affirmative answer, and so Conjecture B is false.

**References**


**University of Wisconsin, Madison**