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ON THE FACTORIZATION OF A CLASS OF DIFFERENCE OPERATORS[1]

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The differential equation for the Meijer G-function (generalized hypergeometric function) with respect to the argument z, [1], can be written in an elegant factored form using the differential operator $z(d/dz)$. Recently, [2], [3], it has been found that particular Meijer G-functions satisfy difference equations with respect to a parameter, and it is the purpose of this paper to deduce analogous factored forms for these difference equations.

Consider the function

$$G(x) = \frac{1}{2\pi i}\int_L z^s \Omega(s) K(s, x, y)\, ds, \tag{1}$$

$$\Omega(s) = \frac{\Gamma(c-s)\prod_{j=1}^{m}\Gamma(b_j - s)\Gamma(1-c+s)\prod_{j=1}^{k}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=k+1}^{p}\Gamma(a_j - s)}, \tag{2}$$

$$0 \leqq m \leqq q, \quad 0 \leqq k \leqq p; \qquad a_j \neq b_i, \quad 1 \leqq j \leqq k, \quad 1 \leqq i \leqq m,$$

$$K(s, x, y) = \Gamma(x+\delta s)/\Gamma(x+y+\epsilon s), \quad \epsilon \text{ and } \delta \text{ integers}, \ \delta \geqq 0, \tag{3}$$

where L is an infinite loop contour which separates the poles of $\Gamma(x+\delta s) \cdot\Gamma(1-c+s)\prod_{j=1}^{k}\Gamma(1-a_j+s)$ from those of $\Gamma(c-s)\prod_{j=1}^{m}\Gamma(b_j-s)$. Here and in what follows, we tacitly assume that the complex quan-

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tities a_i, b_j, c, x, y and z are such that the contour L actually exists. For more details about such integrals, see [1, p. 20].

We define two linear difference operators with respect to x,

$$\mathfrak{A}(\mu, x, y) = \alpha\mathfrak{J} + \beta\mathfrak{E}, \qquad \alpha = (x - \mu\delta)/\Delta, \quad \beta = (\epsilon\mu - x - y)/\Delta,$$

$$\text{(4)} \qquad \mathfrak{A}^*(x, y) = \lim_{\mu\to\infty} \frac{\mathfrak{A}(\mu, x, y)}{\mu} = \alpha^*\mathfrak{J} + \beta^*\mathfrak{E},$$

$$\alpha^* = -\delta/\Delta, \quad \beta^* = \epsilon/\Delta, \quad \Delta = x(\epsilon - \delta) - y\delta \neq 0$$

where $\mathfrak{E}$ is the shift operator $\mathfrak{E}f(x) = f(x+1)$, and $\mathfrak{J}$ is the identity operator. Direct computation shows that

$$\text{(5)} \qquad \begin{aligned} \mathfrak{A}(\mu, x, y)K(s, x, y) &= K(s, x, y + 1)(\mu + s), \\ \mathfrak{A}^*(x, y)K(s, x, y) &= K(s, x, y + 1). \end{aligned}$$

Finally, we set

$$\mathfrak{B} = z\mathfrak{E}^\delta \prod_{j=1}^{p} \mathfrak{A}(1 - a_j, x, y + u + p - j) \prod_{j=1}^{u} \mathfrak{A}^*(x, y + u - j)$$

$$\text{(6)} \qquad + (-1)^{m+p+k} \prod_{j=1}^{q} \mathfrak{A}(-b_j, x, y + v + q - j) \prod_{j=1}^{v} \mathfrak{A}^*(x, y + v - j),$$

$$u = \max\,[0, q - p + \epsilon - \delta], \qquad v = \max\,[0, p - q + \delta - \epsilon].$$

In the ordinary product notation used above, the order of the factors must be interpreted as follows:

$$\prod_{j=1}^{r} P_j = P_1P_2 \cdots P_r.$$

Our principal result is the following

THEOREM. *For the a_i, b_j, c, x, y and z as previously restricted,*

$$\text{(7)} \qquad \mathfrak{B}G(x) = (-1)^{p+k} \frac{z^c\Gamma(x + \delta c)}{\Gamma(x + y + v + q + \epsilon c)} \cdot \frac{\prod_{j=1}^{k} \Gamma(1 + c - a_j) \prod_{j=1}^{m} \Gamma(1 + b_j - c)}{\prod_{j=m+1}^{q} \Gamma(c - b_j) \prod_{j=k+1}^{p} \Gamma(a_j - c)} .$$

PROOF. By applying $\mathfrak{B}$ directly to the integrand of (1), and using (5), together with

$$\Omega(s+1) = \Omega(s)(-1)^{m+k+p+1} \prod_{j=1}^{p} (1-a_j+s) \Big/ \prod_{j=1}^{q} (1-b_j+s), \tag{8}$$

one readily verifies that

$$\begin{aligned} \mathfrak{B}G(x) &= \frac{1}{2\pi i}\int_L z^{s+1}\Omega(s) \prod_{j=1}^{p} (1-a_j+s) K(s, x+\delta, y+u+p) ds \\ &\quad - \frac{1}{2\pi i}\int_{L-1} z^{s+1}\Omega(s) \prod_{j=1}^{p} (1-a_j+s) K(s+1, x, y+v+q) ds. \end{aligned} \tag{9}$$

As $K(s, x+\delta, y+u+p) = K(s+1, x, y+u+p+\delta-\epsilon)$, and $u+p+\delta-\epsilon = v+q$, $\mathfrak{B}G(x)$ is just equal to the sum of the residues of $z^{s+1}\Omega(s) \cdot \prod_{j=1}^{p}(1-a_j+s)K(s+1, x, y+v+q)$ contained in the region between L and $L-1$. By inspection, we see the only possible residue is at $s=c-1$, and (9) reduces to (7).

REMARK 1. It should be noted that there is a certain arbitrariness in the definition of $\mathfrak{B}$, which is attributable to the symmetry property

$$\mathfrak{A}(\mu_2, x, y+1)\mathfrak{A}(\mu_1, x, y) = \mathfrak{A}(\mu_1, x, y+1)\mathfrak{A}(\mu_2, x, y). \tag{10}$$

Clearly, $\mathfrak{B}$ can be rewritten in the form

$$\begin{aligned} \mathfrak{B} &= \sum_{j=0}^{\tau} [A_j + zB_j]\mathfrak{E}^j, \qquad B_0 = 0, \\ \tau &= \max\{q, q+\epsilon, p+\delta, p+\delta-\epsilon\}. \end{aligned} \tag{11}$$

REMARK 2. In reference [3] it was shown that the extended Jacobi functions

$$\begin{aligned} &{}_{r+3}F_t\left(\begin{matrix} -n, n+\lambda, \sigma_r, 1 \\ \rho_t \end{matrix} \middle| z\right) \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} \frac{\prod_{j=1}^{t}\Gamma(\rho_j)}{\prod_{j=1}^{r}\Gamma(\sigma_j)} G_{r+3,t+1}^{1,r+2}\left(z \middle| \begin{matrix} 1-n-\lambda, 1-\sigma_r, 0, n+1 \\ 0, 1-\rho_t \end{matrix}\right) \end{aligned} \tag{12}$$

and the extended Laguerre functions

$$
(13)\qquad {}_{r+2}F_t\left(\begin{matrix}-n, \sigma_r, 1\\ \rho_t\end{matrix}\middle| z\right) = \frac{\Gamma(n+1)\prod_{j=1}^{t}\Gamma(\rho_j)}{\prod_{j=1}^{r}\Gamma(\sigma_j)} G^{1,r+1}_{r+2,t+1}\left(z\middle|\begin{matrix}1-\sigma_r, 0, n+1\\ 0, 1-\rho_t\end{matrix}\right)
$$

satisfy normalized difference equations involving a difference operator of the form (11) with

$$
(14)\qquad \tau = \max[r+2, t]
$$

and

$$
(15)\qquad \tau = \max[r+1, t],
$$

respectively. Furthermore, it was shown that these functions satisfied no other difference equation so normalized of orders $\leqq$ those given by (14) and (15), respectively, provided certain conditions on ρ_i, σ_j, λ were satisfied.

But the G-function on the right in (12) is the integral (1) with

$$
(16)\qquad \begin{matrix} m=0, & k=p=r, & q=t, & c=0, & x=n+\lambda, \\ & y=1-\lambda, & \delta=1, & \epsilon=-1, & \end{matrix}
$$

while the right-hand side of (13) is, apart from a constant multiple, (1) with

$$
(17)\qquad \begin{matrix} m=0, & k=p=r, & q=t, & c=0, & x=n+1, \\ & y=0, & \delta=0, & \epsilon=-1. & \end{matrix}
$$

Furthermore, the formula for τ in (11) gives (14) for the values (16), and (15) for the values (17). In view of the aforementioned uniqueness of the difference equations, it follows that (6) will yield a factorization of those difference equations given in [3].

References

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