$O(h^{2n+2-l})$ BOUNDS ON SOME SPLINE INTERPOLATION ERRORS[1]

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Communicated by R. C. Buck, May 23, 1968

For four or five years it had been felt that C^{2n}, $2n+1$-degree polynominal spline interpolation of a sufficiently smooth function at equally spaced joints (h apart) yielded $O(h^{2n+2-l})$ accuracy in approximating its lth derivative. This was recently shown to be true for periodic boundary conditions on the spline, s, which interpolates $f\in C^{2n+2}(-\infty, \infty)$, f periodic with period 1 ([**1**, p. 151]; [**2**, Theorem 4] as improved by [**3**, last paragraph]). It is shown here that the errors then are the same (up to a higher order term) as the errors associated with local two-point $2n+1$-degree polynomial interpolation, H, of f and its first n odd derivatives at the joints (Theorem 1). The first term in the asymptotic expansion of $\|f^{(l)}-s^{(l)}\|_\infty$ is also derived; it is quite local in character. Theorem 2 states that the same results hold for the spline interpolating $f\in C^{2n+2}[0, 1]$ which matches f and its first n odd derivatives at 0 and 1 as well.

Complete proofs and further references are given in [**3**]. The somewhat less satisfactory situation for another boundary condition (first n *even* derivatives) is also discussed there together with results for other norms and for rougher functions, f. The emphasis in [**3**] is on strict, rather than asymptotic, bounds.

The foundations of the proof arose from consideration of some cubic spline interpolations at arbitrarily spaced joints. Here the techniques yielded some error bounds better than any yet published. They showed further that for $f\in C^4[0, 1]$, $f-s$ is locally bounded by $O(h_i^2h_M^2)$, where h_i is the local mesh size and h_M is the maximum mesh size [**3**].

The typical proof of $O(h^{2n+2-l})$ errors determines first a bound on a high-order derivative of $(f-s)(x)$; then obtains rough bounds on lower-order derivatives by observing that they all have zeroes reasonably nearby. The technique used here, however, is to write $f-s$ as $f-H$ plus the piecewise polynomial $H-s$. $(f^{(l)}-H^{(l)})(x)$ is computed by classical Green's functions arguments. The main lemmas then bound $H^{(l)}-s^{(l)}$ by $O(h^{2n+2-l}\omega(f^{(2n+2)}, h))$. One thus concludes (with the notation $h\equiv 1/N$, $g_i\equiv g(ih)$ for any function g,

Presented to the Society January 23, 1968.

[1] Work performed under auspices of the U. S. Atomic Energy Commission.

$$\|g\|_{\infty,i} \equiv \max_{[ih,(i+1)h]} |g(x)|, \quad \|g\|_\infty \equiv \max_{[0,1]} |g(x)|,$$
$$\omega(g, h) \equiv \sup_{|x-y|\leq h} |g(x) - g(y)|).$$

THEOREM 1. *Suppose f is periodic of period* 1; $f \in C^{2n+2}(-\infty, \infty)$. *Suppose* $N > 2n+1 \geqq 3$. *Let* $s \in C^{2n}(-\infty, \infty)$ *be the* $2n+1$*-degree polynomial spline interpolating f at the joints* $\{ih\}$ *and satisfying periodic boundary conditions at* 0 *and* 1 *as well. Let H, a polynomial of degree* $2n+1$ *between the joints, interpolate* f_i, $f_i^{(2j-1)}$, $1 \leqq j \leqq n$, $0 \leqq i \leqq N$. *Then, for* $0 \leqq l \leqq 2n+1$, *for* $0 \leqq i < N$, *and for* $x \in [ih, (i+1)h]$,

(I) *there exists a Green's function,* $G(s, t)$, *on* $[0, 1] \times [0, 1]$ *such that*

$$(1) \quad f^{(l)}(x) - H^{(l)}(x) = h^{2n+2-l} \int_0^1 \frac{\partial^l G}{\partial s^l}\left(\frac{x - ih}{h}, t\right) f^{(2n+2)}(ih + th)dt;$$

and

(II) *there are constants* $K_{l,n}$ *such that*

$$(2) \quad \|H^{(l)} - s^{(l)}\|_\infty \leqq h^{2n+2-l} K_{l,n} \omega(f^{(2n+2)}, h).$$

Thus, if f is a polynomial of degree $2n+2$, $H \equiv s$.

THEOREM 2. *Suppose, instead, that* $f \in C^{2n+2}[0, 1]$ *and that* $s \in C^{2n}[0, 1]$ *is the* $2n+1$*-degree spline interpolating f at the joints* $\{ih\}$ *while matching the first n odd derivatives of f at* 0 *and* 1 *as well. Then the rest of Theorem* 1 *holds unaltered.*

COROLLARY 1. *For odd* $l < 2n + 1$,

$$\max_i | f_i^{(l)} - s_i^{(l)} | = O[h^{2n+2-l} \omega(f^{(2n+2)}, h)].$$

The same result holds half-way between the joints for odd $l \leqq 2n+1$.

$(f-s)^{(2k)}$ is similarly small at the images (in each interval $[ih, (i+1)h]$) of the two zeroes of the Bernoulli polynomial $B_{2n+2-2k}(x)$; that is, roughly at $(i+1/4)h$ and $(i+3/4)h$ [9].

COROLLARY 2. *For h small enough; and for odd (even) l,* $0 < l \leqq 2n+1$; $f^{(l)} - s^{(l)}$ *vanishes at least once (twice) in all intervals* $(ih, (i+1)h)$ *which are bounded away from the zeroes of* $f^{(2n+2)}$.

COROLLARY 3. *Constants C_k and D_k may be computed such that*

$$\|f - s\|_{\infty,i} = h^{2n+2}[\|f^{(2n+2)}\|_{\infty,i} C_n + O(\omega(f^{(2n+2)}, h))];$$

and, for $1 \leqq l \leqq 2n+1$,

$$\|f^{(l)} - s^{(l)}\|_{\infty,i} = h^{2n+2-l}[\|f^{(2n+2)}\|_{\infty,i} D_{2n+2-l} + O(\omega(f^{(2n+2)}, h))].$$

Indeed,

$$C_k = 2 | B_{2k+2} | (1 - 1/2^{2k+2})/(2k + 2)! < 2D_{2k+2},$$

while

$$D_k = \begin{Bmatrix} |B_k|/k!, & k \text{ even} \\ \|B_k(x)\|_\infty/k!, & k \text{ odd} \end{Bmatrix} < 2/[(2\pi)^k(1-2^{1-k})];$$

where B_k $(B_k(x))$ *is the kth Bernoulli number (polynomial)* [8], [9], *Appendix.*

An outline of the proof now follows.

Existence and uniqueness of s in Theorem 1 is shown in [1] (this also follows from the comment after Lemma 5). As for H: [4] and [5] discuss two-point polynomial interpolation of a function and some of its derivatives. Listed there are those sets of $m+1$ function and derivative values (some at each end) which may be assumed uniquely by a polynomial of degree m. Matching f and its first n odd derivatives at two points by a polynomial of degree $2n+1$ is one of these "Polya conditions." Thus H exists and is unique. Existence of the spline in Theorem 2 now follows. For let P, of degree $2n+1$, interpolate f and its first n odd derivatives at the ends. Set $f^* \equiv f-P$, reflect it evenly in 0, and extend it by periodicity of period 2. Interpolate it at the joints in $[-1, 1]$ by a unique periodic spline, s^*; and set $s \equiv s^*+P$. (Since f^* may only be in $C^{2n}(-\infty, \infty)$, Theorem 2 is not a corollary of Theorem 1.)

Let us turn now to the first part of the error, $f-H$. The existence and uniqueness of the "Polya interpolation" above implies the existence of the Green's function for the boundary-value problem (BV): $y^{(2n+2)} \equiv g$ in $(0, 1)$, y and its first n odd derivatives vanishing at 0 and 1. Using this Green's function, $G(s, t)$, one derives an integral representation for $f-H$ (from which (1) follows):

LEMMA 1. *Suppose* $f \in C^{2n+2}[0, h]$. *Let* H, *of degree* $2n+1$, *interpolate* f *and its first n odd derivatives at* 0 *and* h. *Then, for* $0 \leqq l \leqq 2n+1$,

$$f^{(l)}(x) - H^{(l)}(x) = h^{2n+2-l} \int_0^1 \frac{\partial^l G}{\partial s^l}\left(\frac{x}{h}, t\right) f^{(2n+2)}(ht)\,dt. \tag{3}$$

Let $Q_{2n+2}(s)$ be the $2n+2$-degree polynomial solving the boundary-value problem (BV) with $g \equiv 1$. Then, for $y \in [0, h]$,

$$\begin{aligned} f^{(l)}(x) - H^{(l)}(x) = h^{2n+2-l} \Big\{ & f^{(2n+2)}(y) Q_{2n+2}^{(l)}(x/h) \\ & + \int_0^1 \frac{\partial^l G}{\partial s^l}\left(\frac{x}{h}, t\right) [f^{(2n+2)}(ht) - f^{(2n+2)}(y)]\,dt \Big\}. \end{aligned} \tag{4}$$

From the properties of the Bernoulli polynomials and numbers, $B_k(x)$ and B_k, one sees

$$Q_{2n+2}(x) \equiv (B_{2n+2}(x) - B_{2n+2})/(2n+2)!;$$
$$Q_{2n+2}^{(l)}(x) \equiv B_{2n+2-l}(x)/(2n+2-l)!.$$

(2), (4), and the properties of the $Q_{2n+2}^{(l)}$ now prove Corollaries 1 and 2. But (4) also implies that

$$(5)\quad \|f^{(l)} - H^{(l)}\|_\infty = h^{2n+2-l}[\|f^{(2n+2)}\|_{\infty,0}\|Q_{2n+2}^{(l)}\|_\infty + O(\omega(f^{(2n+2)}, h))],$$

thus proving Corollary 3. Results like (3), (4), and (5) hold for all Polya's boundary conditions [**4**], [**5**].

The remainder of this note concerns the second part of the error, the $2n+1$-degree piecewise-polynomial $H-s$. It vanishes at the joints $\{ih\}$. Bounds on its odd derivatives at the joints are found below, and the following result will be needed:

LEMMA 2. *If bounds, A_{2k-1}/h^{2k-1}, on the first n odd derivatives (at 0 and h) of a $2n+1$-degree polynomial, P, are available; and if $P(0) = P(h) = 0$; then B_l exist such that $\|P^{(l)}\|_{\infty,0} \leqq B_l/h^l$, $0 \leqq l \leqq 2n+1$. These B_l may be calculated by setting $h = 1$ and considering only polynomials which assume the odd derivative bounds at* 0 *and* 1.

For each seminorm $\|P^{(l)}\|_{\infty,0}$ assumes its maximum (over the compact parallelepiped defined by the specified bounds) at a vertex. (Again, the same sort of result holds for Polya's general end conditions [**4**] and [**5**].)

We now turn to estimates of the odd derivatives of $H-s$ at the joints.

Many relations hold between the values of a spline and its derivatives at the joints. The following generalizes [**6**]:

LEMMA 3. *For any spline s, of degree $m \geqq 2$ and in $C^{m-1}[0, 1]$; and for each ν, $0 \leqq \nu \leqq N+1-m$; and for each l, $1 \leqq l \leqq m-1$; there is a linear relation between the m quantities, $s_{j+\nu}$, and the m quantities, $s_{j+\nu}^{(l)}$, $0 \leqq j \leqq m-1$. This relation is given by*

$$(6)\qquad \sum_{j=0}^{m-1} a_j^{(m,l)} s_{j+\nu} = h^l \sum_{j=0}^{m-1} b_j^{(m)} s_{j+\nu}^{(l)}.$$

The coefficients may be written as

$$a_j^{(m,l)} = (-1)^l \sum_{i=0}^{l} (-1)^i \binom{l}{i} Q_{m-l+1}(j+1-i),$$
$$b_j^{(m)} = Q_{m+1}(j+1)$$

where

$$Q_k(x) \equiv \frac{1}{(k-1)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x-i)_+^{k-1}.$$

The proof is a straight-forward generalization of the proof of the result in [6], found in [7, p. 436], and attributed to Schoenberg. For

$$m = 2n+1, \qquad a_{n-j}^{(m,l)} = (-1)^l a_{n+j}^{(m,l)}, \qquad b_{n-j}^{(m)} = b_{n+j}^{(m)} > 0,$$

and

$$m! b_0^{(m)} = 1.$$

One now subtracts $h^l \sum_j b_j^{(m)} f_{j+\nu}^{(l)}$ from both sides of (6), and applies Taylor's theorem with integral remainder to show

LEMMA 4. Let $s \in C^{2n}[0, 1]$ *be a spline of degree* $2n+1$ *interpolating* $f \in C^{2n+2}[0, 1]$ *at the joints* $x_i = ih$. *Define, for* $0 < l < 2n+1$, *and for* $n \leqq i \leqq N-n$,

$$T_l(x_i) \equiv \sum_{j=-n}^{n} b_{n+j}^{(2n+1)} (f_{i+j}^{(l)} - s_{i+j}^{(l)}). \tag{7}$$

Then there are numbers $A_{n,l}$ *such that, for odd* $l < 2n+1$,

$$| T_l(x_i) | \leqq h^{2n+2-l} A_{n,l}\, \omega(f^{(2n+2)}, h). \tag{8}$$

PROOF. For all l, $1 \leqq l < 2n+1 \equiv m$,

$$m!(m-l)! T_l(x_i) = h^{m+1-l} \sum_{j=1}^{n} \int_0^j c_{jl}(t) [f^{(m+1)}(x_i + ht) + (-1)^l f^{(m+1)}(x_i - ht)] dt,$$

where

$$c_{jl}(t) = m! b_{n+j}^{(m)} (j-t)^{m-l} - (m-l)! a_{n+j}^{(m,l)} (j-t)^m.$$

(7) may be written in matrix notation as $(2n+1)!\, T_l \equiv \mathfrak{M}_F(f^{(l)} - s^{(l)})$, where $\mathfrak{M}_F$ is a finite $(N+1-2n)$ by $(N+1)$ segment of the doubly infinite $2n+1$-diagonal Toeplitz matrix $\mathfrak{M} = [m_{ij}]: m_{ij} = (2n+1)!\, b_{n-|i-j|}^{(2n+1)}$, $|i-j| \leqq n$; $m_{ij} = 0$ otherwise. Consider the ring of such $2m+1$-diagonal symmetric (rather than Hermitian) complex matrices $\mathfrak{I}$, all $m \geqq 0$. The correspondence between each $\mathfrak{I}$ and the function $R(z) \equiv \sum_{k=-m}^{m} t_{0,k} z^k$ (set up also by the Fourier transform of $R(e^{ix})$) is an isomorphism of two rings. Each such function may be

factored $R(z)\equiv t_{0,m}\prod_{k=1}^{m}(1/z+r_k+z)$; hence each such $\mathfrak{J}=t_{0,m}\mathfrak{J}_1\cdots\mathfrak{J}_m$, $\mathfrak{J}_k$ tridiagonal with generic row $(\cdots, 0, 1, r_k, 1, 0, \cdots)$. It has been shown a number of times [3] that the r_k associated with $\mathfrak{M}$ are greater than 2. The last lemma one requires, then, is

LEMMA 5. *Let* $v=\mathfrak{M}_F w$, $\mathfrak{M}_F=\mathfrak{J}_{1,F}\mathfrak{J}_{2,F}\cdots\mathfrak{J}_{n,F}$. *Define* $v^{(n)}=w$, $v^{(k-1)}=\mathfrak{J}_{k,F}v^{(k)}$. *Suppose* $\|v^{(k)}\|_\infty$ *occurs elsewhere than either end of* $v^{(k)}$, $1\leq k\leq n$. *Then*

$$\|v\|_\infty \geqq \prod_{k=1}^{n}(r_k-2)\|w\|_\infty$$

$$= R(-1)\|w\|_\infty = (2n+1)!\left[b_n^{(2n+1)}+2\sum_{k=1}^{n}(-1)^k b_{n-k}^{(2n+1)}\right]\|w\|_\infty.$$

Equality holds for v whose components are alternately ± 1.

(One may also show that $\|\mathfrak{M}^{-1}\|_\infty=1/R(-1)$, given by [3, (7.7)].)

Turning to prove (2) of Theorem 1, one first considers $H-s$ on $[-1, 2]$ instead of on $[0, 1]$. One observes that, for $w=f^{(l)}-s^{(l)}$, the existence of the interior maxima for $v^{(k)}$ of Lemma 5 may be verified by periodicity. Thus, by (8), $|H_i^{(l)}-s_i^{(l)}|$ (for odd $l<2n+1$) are uniformly bounded by $(2n+1)!h^{2n+2-l}A_{n,l}\,\omega(f^{(2n+2)}, h)/R(-1)$. Lemma 2 now proves (2).

As for (2) of Theorem 2, one first extends the definition of $H-s$ to $[-1, 1]$ as follows: subtract the $2n+1$-degree Taylor polynomial for f about 0 from f, s, and H; and reflect the results evenly in 0. The resulting f^* is in $C^{2n+2}[-1, 1]$. s^* is a spline interpolating f^*. $H^*-s^*=H-s$ in $[0, 1]$; $H-s$ is thereby extended to $[-1, 1]$. Now do the same thing at 1. The extended $H^{(l)}-s^{(l)}$ still satisfies (8) and (now) the hypotheses of Lemma 5. (2) follows as above.

APPENDIX

The first few D_k and C_k of Corollary 3 are:

$D_5=(1-4/\sqrt{30})^{1/2}(3+\sqrt{30})/21600$, $D_{11}\approx 3.32\ 10^{-9}$, and

k	1	2	3	4	5
D_k	1/2	1/12	$\sqrt{3}/216$	1/6!	above
D_{5+k}	4/(3·8!)	$5.17\ 10^{-6}$	3/10!	$1.31\ 10^{-7}$	10/12!
$(2k+2)!2^{2k+2}C_k$	1	3	17	155	2073

It might be noted that corresponding C_k for two-point *Hermite* interpolation (of f and its first k derivatives) are $1/(2k+2)!/2^{2k+2}$.

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