SCHAUDER DECOMPOSITIONS IN BANACH SPACES[1]

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A sequence (M_i) of closed subspaces of a Banach space E is called a Schauder decomposition of $[M_i]$, the smallest subspace containing $\bigcup M_i$, if every element u of $[M_i]$ has a unique, norm convergent expansion $u=\sum u_i$, where $u_i \in M_i$ for $i=1, 2, \cdots$. It is well known (see, e.g. [2]) that any sequence $(u_i)\subset E$ with $0\neq u_i \in M_i$ for $i=1, 2, \cdots$ is basic (i.e., a basis for its closed linear span). The converse of this statement is not true, but we do derive the following theorem, and mention several corollaries.

THEOREM. *Let (M_i) be a sequence of closed subspace of the Banach space E such that each sequence $(u_i)\subset E$ with $0\neq u_i\in M_i$ is basic. Then there exists an integer N such that $(M_i \mid i\geqq N)$ is a Schauder decomposition of $[M_i \mid i\geqq N]$.*

To simplify the proof of the theorem, we use the following characterization of Schauder decompositions due to Grinblyum [3]. A sequence (M_i) of closed subspaces of E is a Schauder decomposition of $[M_i]$ if and only if there exists a constant K such that for all integers n, m and all sequence (u_i) with $u_i\in M_i$, $\left\| \sum_{i=1}^{n} u_i\right\| \leqq K\left\| \sum_{i=1}^{n+m} u_i\right\|$. We note that this norm condition may be replaced by $\left\| \sum_{i=1}^{n} a_i u_i\right\| \leqq K\left\| \sum_{i=1}^{n+m} a_i u_i\right\|$ where the scalars (a_i) are also arbitrary. Since a sequence $U=(u_i)$ is basic if and only if there exists $K=K(U)$ such that this last inequality holds for all (a_i), m and n, we see that each (u_i) with $u_i\in M_i$ is basic if (M_i) is a Schauder decomposition.

Let $U=(u_i)$ be a sequence with $0\neq u_i\in M_i$, and set $U_n=(u_i \mid i\geqq n)$. Let $K(U_n)$ be the smallest constant such that $\left\| \sum_{i=n}^{p} a_i u_i\right\| \leqq K\left\| \sum_{i=n}^{p+q} a_i u_i\right\|$ holds for all $K\geqq K(U_n)$, all (a_i) and integers p, q.

LEMMA. *Let (M_i) be a sequence of closed subspaces of E such that each $U=(u_i)$ with $0\neq u_i\in M_i$ is basic. Then there exists an integer N and a constant $K\geqq 1$ such that every sequence U as above has $K(U_N)\leqq K$.*

PROOF. If K and N do not exist, then for each integer n and each $M\geqq 1$, there exists a U with $K(U_n)>M$ (noting $K(U_{n+1})\leqq K(U_n)$). Choose $U^{(1)}$ so that $K(U^{(1)})>2$. Then there exist integers $q_1>p_1$ such that $\left\| \sum_{j=1}^{p_1} a_j u_j^{(1)}\right\| > 2\left\| \sum_{j=1}^{q_1} a_j u_j^{(1)}\right\|$ for some sequence (a_i). Similarly, there exist $U^{(2)}$ and $q_2>p_2$ such that

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$$\left\| \sum_{j=q_1+1}^{p_1} a_i u_i^{(2)} \right\| > 4 \left\| \sum_{j=q_1+1}^{q_2} a_i u_i^{(2)} \right\|,$$

and in general we get $U^{(j)}$ and integers p_j, q_j such that $q_{j-1} < p_j < q_j$ and

$$\left\| \sum_{i=q_{j-1}+1}^{p_j} a_i u_i^{(j)} \right\| > 2^j \left\| \sum_{i=q_{j-1}+1}^{q_j} a_i u_i^{(j)} \right\|.$$

With these bounds, the sequence U defined by $u_i = u_i^{(j)}$ if $q_{j-i} < i \leqq q_j$ is not basic, which is a contradiction proving the lemma.

The theorem follows immediately from the lemma and the Grinblyum criterion.

To see that N is in general greater than 1, let E be separable, (x_i) a basic sequence in E such that codim $[x_i] = \infty$ and E_1 a closed subspace of E which is a quasicomplement but not a complement of $[x_i]$ in E. (For a construction of such an E_1 see Gurarii and Kadec [**4**].) If we set $M_1 = E_1$, $M_2 = [x_1]$, $M_3 = [x_2]$, etc., each sequence with just one element in each M_i is basic, but (M_i) is not a Schauder decomposition of E since $M_1 + [x_i] \subsetneq E$. In order to have $N = 1$, then, we must keep $[M_i \mid i < N]$ from being a quasicomplement of $[M_i \mid i \geqq N]$ for each N. In fact, the addition of this hypothesis is also sufficient, for then we see that $[M_i] = M_1 \oplus M_2 \oplus \cdots \oplus M_{N-1} \oplus [M_i \mid i \geqq N]$, and so (M_i) is a Schauder decomposition of $[M_i]$. These corollaries are now immediate. In each, we let U be an arbitrary sequence (u_i) with $0 \neq u_i \in M_i$, and call U a proper sequence.

COROLLARY. *A sequence (M_i) of closed subspaces of E is a Schauder decomposition if and only if* (a) $[M_i] = [M_i \mid i < n] \oplus [M_i \mid i \leqq n]$ *and* (*b*) *each proper sequence U is basic.*

COROLLARY. *The previous corollary holds with* (a) *replaced by* (a′) $[M_i] = M_k \oplus [M_i \mid i \neq k]$ *for each k.*

COROLLARY. *Let (M_i) be a sequence of finite-dimensional subspaces of E. Then (M_i) is a Schauder decomposition if and only if each proper sequence is basic.*

It is easy to see that an N dimensional Banach space F has a basis $(f_i)_{i=1}^N$ such that

$$\left\| \sum_{i=1}^{p} a_i f_i \right\| \leqq N \left\| \sum_{i=1}^{N} a_i f_i \right\|$$

always holds (using, for example, the result of Taylor [5]). The author does not know what the best bound that can replace N in general will be, but it must be greater than 1 (see, e.g. [1]). However, using the last corollary, and the N-bound above, we obtain:

PROPOSITION. *Let* $\dim M_i \leqq N$, $M_i \in E$ *for* $i=1, 2, \cdots$, *and* $[M_i]=E$. *Set* $N_j=\dim M_1+\dim M_2+\cdots+\dim M_j$. *The following are equivalent*

(a) (M_i) *is a Schauder decomposition of* E,

(b) E *has a basis* (x_i) *with* $M_j=[x_i \mid N_{j-1}<i\leqq N_j]$,

(c) *each proper sequence is basic.*

The proof of the proposition is a routine exercise.

PROBLEM. Does the previous result hold with the weaker assumption $\dim M_i<\infty$?

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