A GALOIS PROBLEM FOR MAPPINGS

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1. **Introduction.** A closure space (A, J) consists of a complete lattice A and a closure operator J defined on A. Given two closure spaces (A, J) and (B, K), and a supremum preserving mapping $f: A \to B$, we say that f is continuous if $f^{\Delta}(x)$ is J-closed in A whenever x is K-closed in B, where $f^{\Delta}: B \to A$ is the infimum preserving mapping given by

$$f^{\Delta}(x) = \sup\{z \in A \mid f(z) \leqq x\}.$$

If A is a complete lattice, (B, K) a closure space and $f: A \to B$ a supremum preserving mapping, then $f^{\Delta}Kf$ is the largest closure operator on A which makes f continuous. In fact, given any family X of supremum preserving mappings from A into (B, K) there exists a unique largest closure operator $\Gamma(X)$ on A which makes all the mappings in X continuous. Conversely, we may associate with each closure operator J on A the family $F(J)$ of all continuous supremum preserving mappings from (A, J) into (B, K). It is easily verified that the correspondences $[\Gamma, F]$ establish a Galois connexion between the set of all families of supremum preserving mappings from A into (B, K) and the set of all closure operators on A. We now wish to determine the Galois closed elements for this Galois connexion, that is, we wish to characterize those closure operators J on A and those families X of supremum preserving mappings from A into (B, K) for which $\Gamma F(J) = J$ and $F\Gamma(X) = X$.

2. **The main theorems.** For the Galois connexion described above, the fact that the set of all closure operators on A, ordered pointwise, is a co-atomistic complete lattice may be used to characterize the Galois closed closure operators on A.

THEOREM 1. *Let A be a complete lattice, and (B, K) a closure space. If K is not the indiscrete closure operator, then every closure operator on A is Galois closed. If K is the indiscrete closure on B, then the indiscrete closure is the only Galois closed closure operator on A.*

[1] This work comprised part of the author's dissertation written under Professor Henry H. Crapo at the University of Waterloo, 1968.

In order to characterize the Galois closed families of supremum preserving mappings we introduce the following terminology. A family X of supremum preserving mappings from A into B is called composition closed if the composition γf is in the family X for every mapping f in X and each supremum preserving mapping γ of B into B. A family X of supremum preserving mappings from A into B is called supremum closed if the pointwise supremum of every family of mappings in X is again in X. By a proof analogous to one used by Ore [2] we can prove the following

THEOREM 2. *Let A be a complete lattice and (B, K) a closure space, where B is a complete chain with the discrete closure K. A family X of supremum preserving mappings from A into (B, K) is Galois closed iff X is composition and supremum closed.*

The hypotheses of Theorem 2 may be considerably weakened. We can show that the Galois closed families of mappings are those which are composition and supremum closed, first for B a direct product of complete chains, then for complete sublattices B of direct products of complete chains, and finally, using a theorem due to Raney [4], for B a completely distributive complete lattice, in all cases with the discrete closure on B, to obtain the following theorem.

THEOREM 3. *If A is a complete lattice, B a completely distributive complete lattice with the discrete closure, and X a family of supremum preserving mappings from A into (B, K), then X is Galois closed iff X is composition and supremum closed.*

In a sense Theorem 3 is the best obtainable result in this direction because we have the following theorem.

THEOREM 4. *If A and B are complete lattices, and if A contains a complete sublattice isomorphic to B, then the following conditions are equivalent:*

(1) *The Galois closed subsets of supremum preserving mappings from A into B are precisely those which are composition and supremum closed.*

(2) *B is a completely distributive complete lattice.*

(3) *Every supremum preserving mapping from A into B can be written as a pointwise supremum of a family of two point supremum preserving mappings from A into B, (where a two point mapping is one whose image set contains at most two distinct elements).*

An immediate consequence of the above results is an extension of Ore's main theorem in [2], namely:

THEOREM 5. *If S is an arbitrary set, T a completely distributive complete lattice with the lower order closure relation, and X a family of mappings from S into T, then X is Galois closed iff X is composition and supremum closed.*

As another consequence of these results we can characterize complete distributivity in the following theorem.

THEOREM 6. *For a complete lattice A the following conditions are equivalent:*

(1) *A is completely distributive.*

(2) *The Galois closed subsets of supremum preserving mappings from A into A are precisely those which are composition and supremum closed.*

(3) *Every supremum preserving mapping from A into A can be written as the pointwise supremum of a family of two point supremum preserving mappings from A into A.*

3. **Remarks.** There are of course many associated questions raised by the above work, and which are still unanswered. For example, the Galois connexion described above depends not only on the codomain B, but also on the closure operator K which is defined on B. We have characterized the Galois closed families of supremum preserving mappings from A into (B, K) in cases where K is assumed to be discrete. Is this particular characterization also valid for cases where K is not the discrete closure operator, and if so, what are these cases? In addition, if B is not a completely distributive complete lattice, then composition and supremum closure are not sufficient to characterize the Galois closed families of supremum preserving mappings. What additional conditions are then needed to characterize Galois closure?

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