L^p BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS

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1. In this note we state some results on existence, uniqueness, and a priori estimates, which have been obtained with parabolic singular integral operators as a main tool.

Let $Lu(x, y, t) = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D^\alpha_{x,y} u(x, y, t) - D_t u(x, y, t)$, where $x \in R^n$, $y > 0$, $0 < t < T$. Here $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$, $\alpha_i \geqq 0$ is an integer, $|\alpha| = \alpha_1 + \cdots + \alpha_{n+1}$, $D^\alpha_{x,y} = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_{n+1}^{\alpha_{n+1}}$, $D_t = \partial/\partial t$.

(1.1) DEFINITION. For $\delta \geqq 0$, $\mathfrak{L}_0^{p,2b,1}(R^n \times (\delta, \infty) \times (0, T))$ is the closure of $C_0^\infty(R^{n+1} \times (0, \infty))$ with respect to the norm $\|u\| = \sum_{|\alpha| \leq 2b} \cdot \|D^\alpha_{x,y} u\|_{L^p} + \|D_t u\|_{L^p}$ where the L^p-norms are taken over $R^n \times (\delta, \infty) \times (0, T)$.

(1.2) THEOREM. *Let L be uniformly parabolic in the Petrowsky sense. Assume that the coefficients, a_α, of L are bounded and measurable for $|\alpha| < 2b$ and for $|\alpha| = 2b$, uniformly Hölder continuous in $R^{n+1}_+ \times [0, T]$. For $1 < p < \infty$ there exists a function $u(x, y, t)$ satisfying*

(1.3) *for each $\delta > 0$, $u \in \mathfrak{L}_0^{p,2b,2}(R^n \times (\delta, \infty) \times (0, T))$ and $Lu = 0$ in $R^{n+1}_+ \times (0, T)$*

(1.4) *$D_y^{l+j} u(x, 0, t) = \phi_j(x, t)$ in the sense of $\mathfrak{L}^p_{2b-1-l-j}(S_T)$ where $S_T = R^n \times (0, T)$, $j = 0, \cdots, b-1$, and l is a fixed number satisfying $0 \leqq l \leqq b$.* (1.4) means $\|D_y^{l+j} u(\cdot, y, \cdot) - \phi_j\|_{\mathfrak{L}^p_{2b-1-l-j}(S_T)} \to 0$ as $y \to 0^+$.

In §3 we define $\mathfrak{L}^p_k(S_T)$ and characterize it in terms of spatial derivatives of order $\leqq k$ and a (fractional) time derivative of order $k/2b$ belonging to $L^p(S_T)$. We observe that for $l = 0$ and for $l = b$ Theorem (1.2) is an existence and uniqueness theorem respectively for the Dirichlet and Neumann problems.

We will later state an extension of Theorem (1.2) by replacing (1.4) with a system $\{B_j\}$ of boundary operators

$$B_j(x, t, D_{x,y}) = \sum_{|\beta| \leq r_j} b_{j,\beta(x,t)} D^\beta_{x,y}, \qquad 1 \leqq j \leqq b, \qquad 0 \leqq r_j \leqq 2b - 1.$$

(1.5) DEFINITION. If $k < 2b$ is an integer, $0 < \delta_1, \delta_2 \leqq 1$, a function b defined on $\bar{S}_T$ is in the class $C(k + \delta_1, k/2b + \delta_2)$ if for some $C > 0$,

(i) for $|\alpha| \leqq k$, $D^\alpha_x b$ is bounded, uniformly continuous in $\bar{S}_T$;

(ii) for $|\alpha| = k$, $|D^\alpha_x b(x, t) - D^\alpha_x b(z, t)| \leqq C|x - z|^{\delta_1}$;

(iii) $|b(x, t) - b(x, s)| \leqq C|t - s|^{(k/2b) + \delta_2}$.

(1.6) DEFINITION. $\{B_j\}$ covers L if for some $\delta_0>0$, $B_0>0$ and for

$$H(z, s; x, \tau)$$

$$= \det\left(|x|^{2b} - i\tau)^{(2b-j-r_k)/2b} \oint \frac{B_k^0(z, s; -ix, -i\zeta)(-i\zeta)^{j-1}}{A(z, 0, s; ix, i\zeta) + i\tau} d\zeta\right) \tag{1.7}$$

(i) $H(z, s; x, \tau) \neq 0$ when $\operatorname{Im} \tau > -\delta_0|x|^{2b}$, $(x, \tau) \neq 0$,

(ii) $|H(z, s; x, \tau)| \geqq B_0 > 0$ for $-\delta_0|x|^{2b} < \operatorname{Im} \tau \leqq 0$,

where B_k^0 denotes the principal part of B_k, and with $\alpha' = (\alpha_1, \cdots, \alpha_n, 0)$,

$$A(x, y, t; i\xi, i\eta) = \sum_{|\alpha|=2b} a_\alpha(x, y, t)(i\xi)^{\alpha'}(i\eta)^{\alpha_{n+1}}. \tag{1.8}$$

The contour integrals are taken over a closed curve lying in the lower half ζ-plane, enclosing all roots ζ of $A(z, 0, s; ix, i\zeta)+i\tau=0$ lying there. $H(z, s; x, \tau)$ is the symbol of the matrix of parabolic singular integral operators corresponding to the system $\{B_j\}$, relative to L.

(1.9) THEOREM (EXISTENCE). *If the system $\{B_j\}$ covers L in* (1.2), *and $b_{k\beta}$ is uniformly continuous if $r_k = 2b-1$, while $b_{k\beta} \in C(2b-1-r_k+\epsilon, (2b-1-r_k+\epsilon)/2b)$ if $r_k < 2b-1$, then* (1.2) *holds with* (1.4) *replaced by* (1.4)$'$ $B_j(x, t; D_{x,y})u(x, 0, t) = \phi_j(x, t)$ *in the sense of* $\mathfrak{L}^p_{2b-1-r_k}(S_T)$, $1 \leqq j \leqq b$.

(1.10) THEOREM (UNIQUENESS). *If L, $\{B_j\}$ are as in* (1.9) *and $\psi \in C^\infty(R^{n+1})$ is nonnegative and equals $(|x|^2+y^2)^{1/2}$ for $|x|^2+y^2 \geqq 1$, then the conditions*

(i) *$u(x, y, t)e^{-c\psi(x,y)} \in \mathfrak{L}_0^{p,2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for some $c \geqq 0$ and each $\delta > 0$,*

(ii) *$Lu = 0$, $x \in R^n$, $y > 0$, $0 < t < T$,*

(iii) *$(B_k u)e^{-c\psi} \to 0$ in $\mathfrak{L}^p_{2b-1-r_k}$ as $y \to 0^+$,*

imply that $u(x, y, t) = 0$ for $y > 0$.

Finally we state an a priori estimate for functions in $\mathfrak{L}_0^{p,2b,1} \cdot (R^n \times (0, \infty) \times (0, T))$ with $1 < p < \infty$ and $p \neq 2b+1$. This was done for $p=2$ by Agranovic and Visik in [1] and for p large enough by Solonnikov in [8].

(1.11) DEFINITION. $B_0^{p,\alpha}(S_T)$ is the closure of $C_0^\infty(R_+^{n+1})$ in the norm

$$\|f\|_{B_{p,\alpha}(S_T)} = \|f\|_{L^p(S_T)} + \left(\int_{R^n} \|f(\cdot + h, \cdot) - f\|^p_{L^p(S_T)} \frac{dh}{|h|^{n+\alpha p}}\right)^{1/p}$$
$$+ \left(\int_{R^n} \iint_{0<t,t+h<T} \frac{|f(x, t+h) - f(x, t)|^p}{|h|^{1+\alpha p/2b}} dt\, dh\, dx\right)^{1/p}.$$

(1.12) THEOREM. *If the* L, $\{B_j\}$ *of* (1.2), (1.9) *have respectively coefficients* a_α *bounded and measurable for* $|\alpha|<2b$, *uniformly continuous in* $\bar{S}_T$ *for* $|\alpha|=2b$, *and coefficients* $b_{\beta k}$ *in* $C(2b-r_k-(1/p)+\epsilon, (2b-r_k-(1/p)+\epsilon)/2b)$ *on* $R^n\times[0, T]$, *with in addition, for some* $c>0$,

$$\left| D_x^\alpha b_{\beta,k}(x, t) - D_x^\alpha b_{\beta k}(x, s) \right| \leqq c \left| t - s \right|^{(1-(1/p)+\epsilon)/2b}$$

then there exists μ, $0<\mu\leqq T$, *depending on the bounds of the coefficients of* L, *the modulus of continuity of* a_α *for* $|\alpha|=2b$, *and the parameter of parabolicity, such that for* $p\neq 2b+1$, $1<p<\infty$ *we have for each* $u\in\mathcal{L}_0^{p,2b,1}(R^n\times(0, \infty)\times(0, T))$,

$$\|u\|_{\mathcal{L}_p^{2b,1}(R_+^{n+1}\times(0,\mu))} \leqq C\|Lu\|_{L^p(R_+^{n+1}\times(0,\mu))} + \sum_{k=1}^{b} \|\Lambda^{2b-1-r_k}B_k u(\cdot, 0, \cdot)\|_{B_{p,1-(1/p)}(S_\mu)};$$

Λ^{2b-1-r_k} *is defined in* §3.

2. *A parabolic singular integral operator* (p.s.i.o.) has the form

$$\begin{aligned} Sf(x, t) &= a(x, t)f(x, t) \\ &+ L^p - \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{R^n} K(x, t; x-z, t-s)f(z, s)dzds + Jf(x, t), \end{aligned} \tag{2.1}$$

where

(i) $a(x, t)$ is bounded and uniformly continuous,

(ii) $K(x, t; z, s)=0$ for $s<0$, $K(x, t; \lambda z, \lambda^{2b}s)=\lambda^{-n-2b}K(x, t; z, s)$ for $\lambda>0$, $\int_{R^n}K(x, t; z, 1)dz\equiv 0$; further conditions on K are given in terms of $\mathfrak{F}_z(K(x, t; z, 1))$ (the partial Fourier transform in the z variable), and may be found in [**3**],

(iii) J is in the class $\mathcal{J}(R_+^{n+1})$ of linear operators on $L^p(S_T)$ satisfying (a) $f(x, t)=0$ for $t>s\Rightarrow Jf=0$ for $t>s$, (b) $\|\chi_{(a,a+\epsilon)}J\chi_{(a,a+\epsilon)}f\|_{L^p(R_+^{n+1})} \leqq\omega(\epsilon)\|\chi_{(a,a+\epsilon)}f\|_{L^p(R_+^{n+1})}$ where $\chi_{(a,b)}$ is the characteristic function of $\{(x, t): a<t<b\}$ and $\omega(\epsilon)\to 0$ as $\epsilon\to 0$.

(2.2) DEFINITION. If S has the form (2.1), the *symbol* of S is

$$\sigma(S)(x, t; z, s) \equiv a(x, t) + \lim_{\epsilon\to 0, R\to\infty}\int_\epsilon^R\int_{R^n} K(x, t; w, r)e^{i(w\cdot z+rs)}dw\, dr.$$

The main theorem used here to prove existence (see [**4**] and [**6**]) is:

(2.3) THEOREM. *If $T=(T_{kj})$ is an $N\times N$ matrix of p.s.i.o.'s then T is invertible on each $\Pi_1^N L^p(S_R)$ if for some $\delta_0>0$, $B_0>0$,*

(i) $\det(\sigma(T_{kj})(s, t; z, \zeta))\neq 0$ *for* $(z, \zeta)\neq(0, 0)$, $\operatorname{Im}\zeta>-\delta_0|z|^{2b}$,

(ii) $|\det(\sigma(T_{kj})(x, t; z, \zeta))|\geqq B_0>0$ *for* $|z|=1$, $-\delta_0\leqq\operatorname{Im}\zeta\leqq 0$.

3. **The spaces $\mathcal{L}_k^p(S_T)$.** These are similar to Bessel potential spaces (see [2], [7]). Put $L_0=(-1)^b\Delta^b+D_t$ where Δ is the spatial Laplace operator. Let $\mathfrak{F}\Omega_0(x)=\exp(-|x|^{2b})$, and put

$$\Gamma_0(x, t) = \Omega_0(xt^{-1/2b})t^{-n/2b} \quad \text{if } t > 0, \qquad 0 \text{ elsewhere.}$$

For $k>0$ let $\Lambda^{-k}(x, t)=\Gamma(k/2b)t^{(k/2b)-1}\Gamma_0(x, t)$ ($\Gamma(\cdot)$ is the gamma function). In the spaces $\mathcal{S}'$ of tempered distributions in x, t, $\mathfrak{F}\Lambda^{-k}=(|x|^{2b}-it)^{-k/2b}$, $0<k\leqq 2b$. For $g\in L^p(S_T)$ put $\Lambda^{-k}g=\Lambda^{-k}*g$, and let $\Lambda^0 g=g$.

(3.1) DEFINITION. $\mathcal{L}_k^p(S_T)$, $1<p<\infty$, denotes the space of functions f such that $f=\Lambda^{-k}*g$ for some $g\in L^p(S_T)$. g is unique, and $\|f\|_{\mathcal{L}_k^p(S_T)}=\|g\|_{L^p(S_T)}$ makes $\mathcal{L}_k^p$ into a Banach space.

(3.2) THEOREM. *Let $f\in L^p(S_T)$, $1<p<\infty$. $f\in\mathcal{L}_k^p(S_T)$, where $0<k\leqq 2b$ if and only if $D_x^\alpha f$, $|\alpha|\leqq k$, and $D_t\Lambda^{-2b+k}f\in L^p(S_T)$. Also,*

$$\|f\|_{\mathcal{L}_k^p(S_T)} \backsim \sum_{|\alpha|\leqq k}\|D^\alpha f\|_{L^p(S_T)} + \|D_t\Lambda^{-2b+k}f\|_{L^p(S_T)}.$$

An inverse Λ^k to Λ^{-k} may be defined using differentiation and parabolic singular integrals, and is used in (1.12); the Fourier transform of Λ^k is $(|x|^{2b}-it)^{k/2b}$.

4. **An indication of the methods of proof.** With A given by (1.8), we set

$$\Gamma_{z,\eta,s}(x, y, t) = \mathfrak{F}_{\xi,\nu}(\exp[A(x, \eta, s; i\xi, i\nu)t])(x, y)$$

($\mathfrak{F}_{\xi,\nu}$ denotes the Fourier transform in the variables ξ, ν) and

$$T_j(z, s; x, y, t) = \int_0^t\int_{R^n}\Lambda^{1-j}(x-w, t-r)D_y^{j-1}\Gamma_{z,0,s}(w, y, r)dw\,dr;$$

$y\neq 0$ and $j=1, \cdots, b$. Essentially we smooth y-derivatives in x, t. Using each T_j as a parametrix, we construct (see Chapter IX of [5] and Chapter 3 of [3]) fundamental solutions

$$\Gamma_j(x, y, t; z, \eta, s) = T_j(z, s; x-z, y-\eta, t-s)$$
$$+\int_0^t\int_{R^{n+1}}\Gamma_{w,v,r}(x-w, y-v, t-r)\Phi_j(w, v, r; z, \eta, s)dw\,dv\,dr$$

and set, for $f_j \in L^p(S_T)$, $1 < p < \infty$,

$$u_j(x, y, t) = \int_0^t \int_{R^n} \Gamma_j(x, y, t; z, 0, s) f_j(z, s) dz ds.$$

(4.1) THEOREM. *For each* $\delta > 0$, $u_j \in \mathcal{L}_p^{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ *and* $Lu_j = 0$ *for* $y > 0$. *Moreover if* $|\gamma| = r < 2b$, *there is a constant* C *independent of* y *such that*

$$\| D^\gamma_{x,y} u_j(\cdot, y, \cdot) \|_{\mathcal{L}^p_{2b-1-r}(S_T)} \leqq C \|f\|_{L^p(S_T)}$$

and $L^p - \lim_{y \to 0} \Lambda^{2b-1-r} D^\gamma_{x,y} u_j(x, y, t) = S_{j,\gamma} f_j$ *where* $S_{j,\gamma}$ *is a p.s.i.o. with symbol*

$$-(|x|^{2b} - it)^{(2b-1-r)/2b} (-ix)^\alpha \oint \frac{(-i\zeta)^{l+j-1}}{A(z, 0, s; ix, i\zeta) + it} d\zeta$$

(cf. (1.7), (1.8)).

(4.2) COROLLARY. *Let* u_j *be defined as in* (4.1) *and set* $u(x, y, t) = \sum_{j=1}^b u_j(x, y, t)$. *Assume* L *and* $\{B_j\}$ *satisfy the conditions of* (1.9). *Then for each* $\delta > 0$, $u(x, y, t) \in \mathcal{L}_0^{p,2b,1}(R^n \times (\delta, \infty) \times (0, T))$, $Lu = 0$ *for* $y > 0$ *and* $L^p - \lim_{y \to 0} \Lambda^{2b-1-r_k} [B_k(x, t; D_{x,y}) \, u(x, y, t)] = \sum_{j=1}^b S_{k,j} f_j$, *where* $S_{k,j}$ *is a p.s.i.o. and the matrix* $(\sigma(S_{k,j})(x, t; z, s))_{k,j}$ *is given by* (1.7).

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