

# SPACES DETERMINED BY A GROUP OF FUNCTIONS

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**1. Introduction.** Let  $G_F$  denote the group of all homeomorphisms of the topological space  $F$  onto itself, and let  $G_{F'}$  be similarly defined for a space  $F'$ . If  $G_F$  and  $G_{F'}$  are topologized under the point open topology, and if there is a function from  $G_F$  onto  $G_{F'}$  which is a homeomorphism as well as an algebraic isomorphism then Wechsler [1] has determined a sufficient condition for the spaces  $F$  and  $F'$  to be homeomorphic. Thomas [2] has recently generalized Wechsler's theorem by weakening this condition on the spaces  $F$  and  $F'$ . It is the purpose here to generalize Wechsler's theorem in a different direction by using a group of functions other than a group of homeomorphisms.

**2. Preliminaries.** Most of our notation can be found in [1] and [2]; for reference we include the following. The space  $F$  is  $n$ -homogeneous with respect to a group of functions  $G$  provided for any pair of proper  $n$ -tuples  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_n)$ , there is a  $g$  in  $G$  such that  $g(x_i) = y_i$ ,  $i = 1, \dots, n$ . The space  $F$  is  $\omega$ -homogeneous with respect to a group of functions  $G$  provided it is  $n$ -homogeneous with respect to  $G$  for each positive integer  $n$ .

Let  $G_x = \{f \in G : f(x) = x\}$ . Then  $G_x$  is a subgroup of  $G$  and will be called the *subgroup of the point  $x$* . Furthermore  $G/G_x$  will denote the set of left cosets, and cosets will be written as  $fG_x$ .

We will use the point open topology on  $G$  and will consider  $G/G_x$  to have the topology induced by the natural mapping, that is,  $\nu_x: G \rightarrow G/G_x$  defined by  $\nu_x(h) = hG_x$  is to be continuous so that a set  $U$  is open in  $G/G_x$  if and only if  $\nu_x^{-1}(U)$  is open in  $G$ . All spaces are  $T_2$ .

Our main theorem is as follows:

**THEOREM 1.** *Let  $F$  be a topological space, and let  $G$  denote a group of one-to-one functions from  $F$  onto itself with respect to which  $F$  is  $\omega$ -homogeneous, and let  $F'$  and  $G'$  be similarly defined. Suppose that  $\Phi$  is a homeomorphism from  $G$  onto  $G'$  such that  $\Phi$  is an isomorphism. Then there is a homeomorphism from  $F$  onto  $F'$ .*

The proof of the main theorem will be accomplished by showing the existence of a sequence of homeomorphisms whose composition will then be the desired homeomorphism between  $F$  and  $F'$ . We prove first that  $G/G_x$  is homeomorphic to  $F$ . We then show that  $\Phi$  induces a homeomorphism from  $G/G_x$  onto  $G'/\Phi(G_x)$ . It is next shown that the

subgroup  $\Phi(G_x)$  is, in fact, the subgroup of a point  $y$  in  $F'$ , i.e.  $\Phi(G_x) = G'_y$ . We have then that  $G'/\Phi(G_x) = G'/G'_y$ . Finally, an application of the first result gives that  $G'/G'_y$  is homeomorphic to  $F'$ . Thus, letting  $\simeq$  denote the relation of homeomorphism, we have

$$F \simeq G/G_x \simeq G'/\Phi(G_x) = G'/G'_y \simeq F'$$

and therefore  $F$  is homeomorphic to  $F'$ .

**3. The sequence of homeomorphisms.** A subbasis element in  $G$  is denoted by  $W(y, U) = \{f \in G : f(y) \in U\}$  where  $y$  is a point of  $F$  and  $U$  is an open set in  $F$ . A basis element is denoted by  $W(\bar{y}, \bar{U})$  where  $\bar{y} = (y_1, \dots, y_n)$  is a proper  $n$ -tuple and  $\bar{U} = (U_1, \dots, U_n)$ . The symbol  $W(y)$  is used to denote  $\{z \in F : g(y) = z, g \in W\}$ .

**LEMMA 1.** *The function  $\mu_x : G \rightarrow F$  defined by  $\mu_x(h) = h(x)$  is continuous and open.*

**PROOF.** This is shown in [1] for a group  $G$  of homeomorphisms. It is easily seen that the continuity and openness of elements of  $G$  are not required.

**THEOREM 2.** *The function  $\theta_x : G/G_x \rightarrow F$  defined by  $\theta_x(hG_x) = h(x)$  is a homeomorphism.*

**PROOF.** The comment in the proof of Lemma 1 applies here also.

Suppose  $F'$  is a Hausdorff topological space which is  $\omega$ -homogeneous with respect to a group  $G'$  of one-to-one functions from  $F'$  onto itself. Suppose further that there is a homeomorphism  $\Phi$  from  $G$  to  $G'$  which is an isomorphism. Let  $H = G_x$  and  $H' = \Phi(G_x)$ .

**LEMMA 2.** *Using the notation of the previous paragraph,  $\Phi$  induces a homeomorphism between  $G/H$  and  $G'/H'$  defined as  $\phi = \nu'\Phi\nu_x^{-1}$ , where  $\nu'$  is the natural map from  $G'$  onto  $G'/H'$ .*

**PROOF.** Trivial.

Our next task is to show that  $H'$  is the subgroup of a point  $y \in F'$ . For this we will need several lemmas.

**LEMMA 3.** *If  $f \in G - H$ , then  $HfH$  is dense in  $G$ .*

**PROOF.** The proof is contained in Theorem 3.1 of [2] where again the continuity and openness of elements of  $G$  are not required.

**LEMMA 4.**  *$H'$  is a proper closed subgroup of  $G'$ .*

**PROOF.** Trivial.

Throughout the remainder  $W(\bar{x}, \bar{U})$  will denote a nonempty basic open set in  $G' - H'$  such that  $\bar{x} \in F'^n$ ,  $\bar{U} \subset F'^n$  and  $n$  is the smallest

integer with this property. If  $\bar{w} = (w_1, \dots, w_n)$  denote  $(w_1, \dots, w_{p-1}, w_{p+1}, \dots, w_n)$  by  $\bar{w}_p$ .

LEMMA 5. *If  $n > 1$  then  $H'(\bar{x}_k)$  is dense in  $F'^{n-1}$  and therefore infinite in each component.*

PROOF. If there is an open set  $O \subset F'^{n-1}$  which does not intersect  $H'(\bar{x}_k)$ , then  $W(\bar{x}_k, O)$  contradicts the minimality of  $n$ .

LEMMA 6. *If  $n > 1$  there exist two independent points in  $H'(\bar{x})$ .*

PROOF. The proof is as in Lemma 3.14 of [1].

LEMMA 7. *Suppose there is an  $n$ -tuple  $\bar{z}$  in  $H'(\bar{x})$  independent of  $\bar{x}$ . Then  $\bar{x}$  lies in infinitely many distinct sets of the form  $g(H'(\bar{x}))$  where  $g$  is in  $G'$ .*

PROOF. This proof is obtained in the same manner as the similar result in [2, Lemma 2.4].

THEOREM 3. *For some  $y$  in  $F'$ ,  $H' = G'_y$ .*

PROOF. Having established the preceding Lemmas, the proof given in [2, Theorem 3.1] applies.

**4. Proof of Theorem 1.** From Theorem 2, we have that  $F$  and  $G/G_x$  are homeomorphic and so also are  $F'$  and  $G'/G'_y$ . That  $G/G_x$  and  $G'/\Phi(G_x)$  are homeomorphic is a consequence of Lemma 2. Theorem 3 establishes the equality of  $G'/\Phi(G_x)$  and  $G'/G'_y$ . Thus

$$F \simeq G/G_x \simeq G'/\Phi(G_x) = G'/G'_y \simeq F'$$

and therefore  $F$  and  $F'$  are homeomorphic.

#### REFERENCES

1. M. T. Wechsler, *Homeomorphism groups of certain topological spaces*, Ann. of Math. **62** (1955), 360–373.
2. E. S. Thomas, Jr., *Spaces determined by their homeomorphism groups*, Trans. Amer. Math. Soc. **126** (1967), 244–250.

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