1. Introduction. Suppose $Q^{n+1}$ is a piecewise linear $(n+1)$-manifold and $M^n$ is a closed topological $n$-manifold embedded in int $Q^{n+1}$. We seek conditions on the embedding of $M$ which insure that $M$ has arbitrarily small neighborhoods which look like regular neighborhoods of a piecewise linear (PL) submanifold of $Q$. In particular, we would like $M$ to be contained in a compact $(n+1)$-dimensional PL submanifold $N$ of $Q$ such that

1. $M \subset \text{int } N$,
2. $M$ is a strong deformation retract of $N$, and
3. $N - M$ is PL homeomorphic to $\text{bd } N \times [0, 1)$.

We call any compact (connected) PL submanifold $N$ of $Q$ satisfying (1) a PL manifold neighborhood of $M$.

We say that $Q - M$ is 1-lc at $M$ if for each open set $U$ containing $M$ there is an open set $V$, $M \subset V \subset U$, such that each loop in $V - M$ is null homotopic in $U - M$. The purpose of this note is to show that, if $M$ is simply connected and $n \geq 5$, then $M$ has PL manifold neighborhoods satisfying (2) and (3) above if and only if $Q - M$ is 1-lc at $M$.

All homology and cohomology groups will be singular with $\mathbb{Z}$ coefficients. $i_\#$ ($i^*$) will denote an inclusion induced map between homology or homotopy (cohomology) groups. The symbol $\approx$ means is isomorphic to or is PL homeomorphic to, depending on the context. $I$ denotes the unit interval $[0, 1]$.

2. Statement of results. Let $Q^{n+1}$ be a connected PL $(n+1)$-manifold, $M^n$ a closed, 1-connected topological $n$-manifold embedded in int $Q$. Our main result is

**Theorem 1.** If $n \geq 5$, there is a closed PL $n$-manifold $M^*$ such that $M$ has arbitrarily small PL manifold neighborhoods which are PL homeomorphic to $M^* \times I$ and satisfy (2) and (3) above if and only if $Q - M$ is 1-lc at $M$.

The proof is postponed until §3.
Now if $M$ is a PL submanifold of $Q$, $M$ is locally flat, and thus $Q - M$ is 1-lc at $M$. It follows that the boundary components of any regular neighborhood of $M$ are simply connected, so Theorem 1 and the $h$-cobordism theorem [4] (or Lemma 1 below) easily imply

**Corollary 1.** If $M^n, n \geq 5$, is a closed, 1-connected PL submanifold in the interior of a PL $(n+1)$-manifold, regular neighborhoods of $M$ are PL homeomorphic to the product of some closed PL $n$-manifold with $I$.

Corollary 1 generalizes a theorem of Husch, [3].

**Corollary 2.** If $M^n$ is a closed, 1-connected, topological $n$-manifold, $n \geq 5$, which can be embedded in the interior of a PL manifold $Q^{n+1}$ in such a way that $Q - M$ is 1-lc at $M$, then $M$ has the homotopy type of a closed PL $n$-manifold.

3. **Proof of Theorem 1.** We will need the following two lemmas.

**Lemma 1.** If $W^{n+1}$ is a compact PL $(n+1)$-manifold, $n \geq 5$, with exactly two boundary components $A$ and $B$, both 1-connected, and $W$ has the homotopy type of a closed, 1-connected $n$-manifold, then $W \approx A \times I$.

**Proof.** By the $h$-cobordism theorem, it suffices to show that the inclusion $A \subset W$ is a homotopy equivalence, and for this it is sufficient, by a theorem of Whitehead [7], to show that $H_q(W, A) \approx 0$ for all $q$. Since $W$ is 1-connected, $H^1(W, B) \approx 0$, so by Poincaré duality, $H_n(W, A) \approx 0$. Since $H_n(A) \approx H_n(W) \approx \mathbb{Z}$, the exact homology sequence of $(W, A)$ shows that $i_*: H_n(A) \rightarrow H_n(W)$ is an isomorphism, so it follows from [6] that for each $q$, $i_*: H_q(A) \rightarrow H_q(W)$ is onto and $i^*: H^q(W) \rightarrow H^q(A)$ is 1-1. The exact homology and cohomology sequences of $(W, A)$ show that for each $q$,

$$H_{q+1}(W, A) \approx \ker \{ i_*: H_q(A) \rightarrow H_q(W) \},$$

and

$$H^{q+1}(W, A) \approx \text{coker} \{ i^*: H^q(W) \rightarrow H^q(A) \}.$$

By Poincaré duality for kernels [1], [6], there is an isomorphism $\rho_q: H_{q+1}(W, A) \rightarrow H^{n+1-q}(W, A)$ for each $q$. Composing $\rho_q$ with the Poincaré duality isomorphism for $(W, A, B)$ gives an isomorphism $H_{q+1}(W, A) \approx H^{n+1-q}(W, A) \approx H_q(W, B)$ and a similar argument gives $H_{q+1}(W, B) \approx H_q(W, A)$.

Since $H_0(W, A) \approx H_0(W, B) \approx 0$, it follows that $H_q(W, A) \approx 0$ for each $q$, and the proof is complete.

**Lemma 2.** If $M^n, Q^{n+1}$ are as in Theorem 1 and $Q - M$ is 1-lc at $M$, then $M$ has arbitrarily small PL manifold neighborhoods $W$ such that
(a) $W$ has exactly two boundary components, each 1-connected,
(b) $W$ is 1-connected, and
(c) $W - M$ has two components, each 1-connected.

**Proof.** We may assume that $Q$ is open and 1-connected, for if $\tilde{Q}$, int $Q$ is the (PL) universal cover of int $Q$, the fact that $M$ is 1-connected implies that $p$ is a homeomorphism on each component of $p^{-1}(M)$. If $\tilde{M}$ is a component of $p^{-1}(M)$, a standard compactness argument shows that $p$ is a homeomorphism on some neighborhood of $\tilde{M}$, so we may replace $Q$ and $M$ by $\tilde{Q}$ and $\tilde{M}$.

Now Alexander duality [5, Theorem 6.2.17] shows that $H_1(Q, Q - M) \approx Z$, and the reduced homology sequence of $(Q, Q - M)$ shows that $Q - M$ has exactly two components, say $Q_1$ and $Q_2$. Furthermore, it is easy to see that if $U$ is any connected open neighborhood of $M$, $U - M$ has exactly two components, $Q_i \cap U$, $i = 1, 2$.

Let $U$ be any open neighborhood of $M$, and let $V$ be a neighborhood of $M$ such that $V \subseteq U$ and each loop in $V - M$ is null homotopic in $U - M$. We may also assume that each loop in $U - M$ is null homotopic in $Q - M$. Let $W_1$ be any PL manifold neighborhood of $M$ in $V$, and suppose $W_1$ has more than one boundary component in $Q_i$. If $A$ and $B$ are two of these components, we may join them with a polyhedral arc $\alpha$ which lies, except for its endpoints, in int $W_1 - M$. If $T$ is a small regular neighborhood of $\alpha$ in $W_1 - M$, $\text{Cl}(W_1 - T)$ is a PL manifold neighborhood of $M$ in $V$ with one less boundary component in $Q_1$. Continuing this process on both sides of $M$, we can find a PL manifold neighborhood $W_2$ of $M$ in $V$ which has exactly one boundary component in each of $Q_1$ and $Q_2$. Since each loop in bd $W_2$ is null homotopic in $U - M$ by our choice of $V$, and since $n \geq 5$, we may alter $W_2$ by “exchanging disks” [1] to get a PL manifold neighborhood $W$ of $M$ in $V$ which satisfies (a). By two applications of the Van Kampen theorem, $W$ is 1-connected, so (b) holds. $W - M$ has two components, and if $N_j$ is the component contained in $Q_j$, $j = 1, 2$, our assumption on $U$ implies that $i_*: \pi_1(N_j) \to \pi_1(Q_i)$ is trivial, so the Van Kampen theorem implies that $N_j$ is 1-connected, and the proof is complete.

**Proof of Theorem 1.** Necessity is obvious. For the converse, let $W$ be one of the PL manifold neighborhoods guaranteed by Lemma 2, small enough that $M$ is a retract of $W$. Let $V$ be the double of $W$ and think of $V$ as $W_1 \cup W_2$ where the $W_i$ are the two copies of $W$ and $M \subseteq W_1$. Then $V - M$ is an open PL manifold, and Lemma 2 shows that $V - M$ has two simply connected ends. Alexander duality and the exact homology sequence of $(V, V - M)$ show that the
homology of $V-M$ is finitely generated, so by the Browder, Levine, and Livesay boundary theorem [2], $V-M$ is PL homeomorphic to the interior of a compact PL manifold with two simply connected boundary components. This means that there is a compact, PL submanifold $X^{n+1}$ of $V-M$ such that $X$ has two simply connected boundary components, $W_2 \subset \text{int} \, X$, and $(V-M) - \text{int} \, X \approx \text{bd} \, X \times [0, 1)$. Let $N = V - \text{int} \, X$. Then $N$ clearly satisfies conditions (1) and (3) of the introduction. Since $V$ is orientable, Alexander duality gives $H^q(N, M) \approx H_{n+1-q}(V-M, \text{int} \, X) \approx 0$ for each $q$. By the Universal Coefficient Theorem, $H_q(N, M) \approx 0$ for each $q$. $N$ is simply connected by the arguments of Lemma 2, so by the Whitehead theorem, $i : M \rightarrow N$ is a homotopy equivalence. Since $N$ retracts onto $M$, it follows that $M$ is a strong deformation retract of $N$, so $N$ satisfies (3). $N$ is a product by Lemma 1, so the proof is complete.

References


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