

HOMOTOPY-EVERYTHING H -SPACES

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An H -space is a topological space X with basepoint e and a *multiplication* map $m: X^2 = X \times X \rightarrow X$ such that e is a homotopy identity element. (We take all maps and homotopies in the based sense. We use k -topologies throughout in order to avoid spurious topological difficulties. This gives function spaces a canonical topology.) We call X a *monoid* if m is associative and e is a strict identity.

In the literature there are many kinds of H -space: homotopy-associative, homotopy-commutative, A_∞ -spaces [3], etc. In the last case part of the structure consists of higher *coherence* homotopies. In this note we introduce the concept of *homotopy-everything H -space* (E -space for short), in which all possible coherence conditions hold. We can also define E -maps (see §4). Our two main theorems are Theorem A, which classifies E -spaces, and Theorem C, which provides familiar examples such as BPL . Many of the results are folk theorems. Full details will appear elsewhere.

A space X is called an *infinite loop space* if there is a sequence of spaces X_n and homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for $n \geq 0$, such that $X = X_0$.

THEOREM A. *A CW-complex X admits an E -space structure with $\pi_0(X)$ a group if and only if it is an infinite loop space. Every E -space X has a "classifying space" BX , which is again an E -space.*

1. The machine. This constructs numerous E -spaces.

Consider the category \mathcal{g} of real inner-product spaces of countable (algebraic) dimension and linear isometric maps between them. As examples we have \mathbf{R}^∞ with orthonormal base $\{e_1, e_2, e_3, \dots\}$, and its subspace \mathbf{R}^n with base $\{e_1, e_2, \dots, e_n\}$, which is all there are up to isomorphism. We topologize $\mathcal{g}(A, B)$, the set of all isometric linear maps from A to B , by first giving A and B the *finite* topology, which makes each the topological direct limit of its finite-dimensional subspaces.

LEMMA. *The space $\mathcal{g}(A, \mathbf{R}^\infty)$ is contractible.*

This is a consequence of two easily constructed homotopies:

- (a) $i_1 \simeq i_2: A \rightarrow A \oplus A$,
- (b) $i_1 \simeq u: \mathbf{R}^\infty \rightarrow \mathbf{R}^\infty \oplus \mathbf{R}^\infty$, for some isomorphism u .

Suppose we have a functor T defined on the category \mathcal{S} , taking topological spaces as values, and a continuous natural transformation $\omega: TA \times TB \rightarrow T(A \oplus B)$ called *Whitney sum*, such that:

- (a) Tf is a continuous function of $f \in \mathcal{S}(A, B)$;
- (b) TR^0 consists of one point;
- (c) ω preserves associativity, commutativity and units;
- (d) TR^∞ is the direct limit of the spaces TR^n .

THEOREM B. *TR^∞ is an E-space. If T happens to be monoid-valued, the classifying space [2] BTR^∞ agrees with that from Theorem A.*

As a (noncanonical) multiplication on TR^∞ we take

$$TR^\infty \times TR^\infty \xrightarrow[\omega]{Tf} T(R^\infty \oplus R^\infty) \xrightarrow{Tf} TR^\infty,$$

where $f: R^\infty \oplus R^\infty \rightarrow R^\infty$ is any linear isometric embedding. The Lemma provides homotopy-associativity, since $f \circ (f \oplus 1) \simeq f \circ (1 \oplus f)$, homotopy-commutativity, and all higher coherence homotopies.

In the examples below we define TA explicitly only for finite-dimensional A , and note that axiom (d) extends the definition to the whole of \mathcal{S} . In each case the maps Tf and the Whitney sum ω are obvious, in view of the inner products.

- EXAMPLES.**
1. $TA = O(A)$, the orthogonal group of A . Then $TR^\infty = O$.
 2. $TA = U(A \otimes C)$, the unitary group of $A \otimes C$. Then $TR^\infty = U$.
 3. $TA = BO(A)$, a suitable classifying space for $O(A)$, for example that given by [2]. Then $TR^\infty = BO$.
 4. $TA = F(A)$, the space of based homotopy equivalences of the sphere S^A , which is the one-point compactification $A \cup^\infty$ of A , with ∞ as basepoint. Then $TR^\infty = F$.

There is also a semisimplicial analogue, in which T takes semisimplicial values and $\mathcal{S}(A, B)$ is replaced by its singular complex.

5. $TA = \text{Top}(A)$. A k -simplex of $\text{Top}(A)$ is a fibre-preserving homeomorphism of $A \times \Delta^k$ over Δ^k , where Δ^k is the standard k -simplex. Then $TR^\infty = \text{Top}$.

6. The semisimplicial analogues of Examples 1–4.
7. The orientation-preserving versions of Examples 1–6.
8. $TA = PL(A)$, defined as $\text{Top}(A)$ except taking only piecewise linear homeomorphisms of $A \times \Delta^k$. This *fails*. To cure this we need a new machine. Suffice it to say that as a k -simplex of $\mathcal{O}(A, B)$ we take a pair (ξ, f) , where ξ is a p. l. subbundle of the product bundle $B \times \Delta^k$ over Δ^k , and $f: \xi \oplus (A \times \Delta^k) \cong B \times \Delta^k$ is a p. l. fibrewise homeomorphism that extends the inclusion of ξ .

THEOREM C. *We have the E-spaces $O, SO, F, U, PL, \text{Top}, \text{etc.}$, their coset spaces $F/PL, \Gamma = "PL/O"$ etc., and all their iterated classifying spaces. The natural maps between these are all E-maps, including $O \rightarrow PL$ and $PL \rightarrow \Gamma$.*

2. Categories of operators. There are two variants: *with* or *without* permutations.

DEFINITION. In a category of operators \mathfrak{B}

- (a) the objects are $0, 1, 2, \dots$;
- (b) the morphisms from m to n form a topological space $\mathfrak{B}(m, n)$, and composition is continuous;
- (c) we are given a strictly associative continuous functor $\oplus : \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{B}$ such that $m \oplus n = m + n$;
- (d) if \mathfrak{B} has permutations, we are also given for each n a homomorphism $S_n \rightarrow \mathfrak{B}(n, n)$, S_n the symmetric group on n letters. (We omit any symbol for this homomorphism.) In the case with permutations we impose two further axioms:

(i) if $\pi \in S_m$ and $\rho \in S_n$ then $\pi \oplus \rho$ lies in S_{m+n} and is the usual sum permutation;

(ii) given any r morphisms $\alpha_i : m_i \rightarrow n_i$ and $\pi \in S_r$, we have

$$\pi(\mathbf{n}) \circ (\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_r) = \pi(\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_r) \circ \pi(\mathbf{m}),$$

where $m = \sum m_i, n = \sum n_i, \pi$ permutes the factors of $\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_r$, and the permutation $\pi(\mathbf{n}) \in S_n$ is obtained from π by replacing i by a block of n_i elements. We require functors to preserve all this structure.

EXAMPLES. 1. End_X , for a based space X . $\text{End}_X(m, n)$ is the space of all (based) maps $X^m \rightarrow X^n$, where X^n is the n th power of X . The functor \oplus is just \times . This example has permutations.

DEFINITION. The category \mathfrak{B} of operators *acts on* X , or X is a \mathfrak{B} -space, if we are given a functor $\mathfrak{B} \rightarrow \text{End}_X$.

2. \mathfrak{G} . $\mathfrak{G}(m, n)$ is the set of all order-preserving maps $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$. Then an \mathfrak{G} -space is a monoid.

3. \mathfrak{S} . $\mathfrak{S}(m, n)$ is the set of all maps $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$, including permutations. Then an \mathfrak{S} -space is an abelian monoid.

DEFINITION. We call X an *E-space* if we are given a category \mathfrak{B} of operators with permutations acting on X , for which $\mathfrak{B}(n, 1)$ is contractible for all n . (We do not single out any canonical \mathfrak{B} .)

4. \mathcal{G} . Define $\mathcal{G}(m, n) = \mathcal{G}((\mathbb{R}^\infty)^m, (\mathbb{R}^\infty)^n)$ as in §1. By the Lemma any \mathcal{G} -space, for instance $T\mathbb{R}^\infty$, is an *E-space*.

5. \mathbb{Q}_n , a category of operators on the n th loop space $X = \Omega^n Y$, the space of all maps $(I^n, \partial I^n) \rightarrow (Y, o)$, where I^n is the standard n -cube, ∂I^n its boundary, and o the basepoint of Y . A point $\alpha \in \mathbb{Q}_n(k, 1)$ is a

collection of k n -cubes I_i^n linearly embedded in I^n , with disjoint interiors, and with axes parallel to those of I^n . It acts on X as follows: given $(f_1, f_2, \dots, f_k) \in X^k$, the map $\alpha(f_1, f_2, \dots, f_k): I^n \rightarrow Y$ is given by f_i on each little cube I_i^n , and zero elsewhere. Similarly for $\mathbb{Q}_n(k, r)$. We topologize $\mathbb{Q}_n(k, 1)$ as a subspace of \mathbb{R}^{2kn} . We observe that $\mathbb{Q}_n(k, 1)$ is $(n-2)$ -connected, so that as n tends to ∞ , Theorem A becomes plausible.

We say the category \mathfrak{B} of operators, without permutations, is *in standard form* if every morphism $\alpha: m \rightarrow n$ has uniquely the form $\alpha_1 \oplus \alpha_2 \oplus \dots \oplus \alpha_n$, where $\alpha_i: m_i \rightarrow 1$. For categories with permutations the definition is more complicated. Of our examples, 2, 3 and 5 are in standard form, but 1 and 4 are not.

3. The bar construction. Suppose given a category \mathfrak{B} of operators, in standard form. We consider words $[\alpha_0 | \alpha_1 | \dots | \alpha_k]$, where $k \geq 0$ and each α_i is a morphism in \mathfrak{B} and $\alpha_0 \circ \alpha_1 \circ \dots \circ \alpha_k$ exists.

DEFINITION. The category $W^0\mathfrak{B}$ has as morphisms from m to n those words $[\alpha_0 | \alpha_1 | \dots | \alpha_k]$ for which the composite exists and is in $\mathfrak{B}(m, n)$, subject to the relations and their consequences:

$$[\alpha \oplus \beta] = [\alpha \oplus 1 | 1 \oplus \beta] = [1 \oplus \beta | \alpha \oplus 1] \text{ for appropriate identities } 1;$$

$$[1] \text{ is an identity in } W^0\mathfrak{B};$$

$[\alpha | \pi] = [\alpha \circ \pi]$ and $[\pi | \beta] = [\pi \circ \beta]$ if \mathfrak{B} has permutations π . Composition in $W^0\mathfrak{B}$ is by juxtaposition.

To form the category $W\mathfrak{B}$, we take for each morphism x in $W^0\mathfrak{B}$ a cube $C(x)$ of suitable dimension, having x as vertex, and identify the faces not containing x with certain cubes $C(x_i)$ of lower dimension, where x_i runs through the words formed from x by one "amalgamation." The categories $W^0\mathfrak{B}$ and $W\mathfrak{B}$ inherit obvious identification topologies. For composition we have $C(x) \circ C(y) \subset C(x \circ y)$ as a face containing $x \circ y$, and $\oplus: C(x) \times C(y) \cong C(x \oplus y)$. The *augmentation* $\epsilon: W\mathfrak{B} \rightarrow \mathfrak{B}$ is defined by $\epsilon[\alpha_0 | \alpha_1 | \dots | \alpha_k] = \alpha_0 \circ \alpha_1 \circ \dots \circ \alpha_k$ and $\epsilon C(x) = \epsilon x$.

A $W\mathfrak{A}$ -space, with \mathfrak{A} as in §2, is approximately an A_∞ -space [3]. In particular, the familiar pentagon in $W\mathfrak{A}(4, 1)$ is now subdivided into five squares.

For the following theorems we need a slightly different category \mathfrak{B}' augmented over \mathfrak{B} , in which $\mathfrak{B}'(1, 1)$ differs from $\mathfrak{B}(1, 1)$ by a whisker. However, we may replace \mathfrak{B}' by \mathfrak{B} in the theorems whenever the identity 1 in $\mathfrak{B}(1, 1)$ is an isolated point.

We call an augmentation functor $\theta: \mathfrak{C} \rightarrow \mathfrak{B}$ *fibre-homotopically trivial* if for each n there exists a section $\chi: \mathfrak{B}(n, 1) \rightarrow \mathfrak{C}(n, 1)$ such that $\chi \circ \theta$ is fibrewise homotopic to the identity map of $\mathfrak{C}(n, 1)$, S_n -equivariantly if \mathfrak{B} and \mathfrak{C} have permutations.

THEOREM D. (a) $\epsilon: W\mathcal{B}' \rightarrow \mathcal{B}' \rightarrow \mathcal{B}$ is fibre-homotopically trivial.

(b) Given any category of operators \mathcal{C} augmented over \mathcal{B} by a fibre-homotopically trivial functor, there exists a functor $W\mathcal{B}' \rightarrow \mathcal{C}$ that lifts $W\mathcal{B}' \rightarrow \mathcal{B}$. (It is not unique.)

THEOREM E. Suppose X and Y have the same homotopy type, and $W\mathcal{B}'$ acts on X . Then we can make $W\mathcal{B}'$ act on Y .

4. Maps between H -spaces. Suppose X and Y are $W\mathcal{B}$ -spaces and $f: X \rightarrow Y$ a map. We call f a $W\mathcal{B}$ -homomorphism if it commutes with the action of $W\mathcal{B}$. We need a weaker homotopy notion.

Let \mathcal{L}_n be the "linear" category with objects $0, 1, 2, \dots, n$ and one morphism $i \rightarrow j$ whenever $i \leq j$. We can generalize the bar construction in §3 to form $W(\mathcal{B} \times \mathcal{L}_n)$, a category which we make act on $(n+1)$ -tuples of spaces. (Note that in $\mathcal{B} \times \mathcal{L}_n \oplus$ is not quite a functor, because it is not everywhere defined.) We define a homotopy \mathcal{B} -map from X to Y as an action of $W(\mathcal{B} \times \mathcal{L}_1)$ on the pair (X, Y) , that induces the given $W\mathcal{B}$ -structures on X and Y . Similarly, an action on E -spaces X and Y of a suitable category \mathcal{C} such that $\mathcal{C}(X^n, Y)$ is contractible for all n is called an E -map.

THEOREM F. Let X and Y be $W\mathcal{B}'$ -spaces, and $f: X \rightarrow Y$ a homotopy \mathcal{B}' -map that is also a homotopy equivalence. Then any homotopy inverse $g: Y \rightarrow X$ admits the structure of homotopy \mathcal{B}' -map.

We cannot form the category of $W\mathcal{B}$ -spaces and homotopy \mathcal{B} -maps, because unless one of them is a $W\mathcal{B}$ -homomorphism, the composite of two homotopy \mathcal{B} -maps is defined only up to a homotopy, which is defined only up to a homotopy, which is Instead we form a semi-simplicial complex K , whose n -simplexes are actions of $W(\mathcal{B} \times \mathcal{L}_n)$ on $(n+1)$ -tuples of spaces; in particular its vertices are $W\mathcal{B}$ -spaces and its edges are homotopy \mathcal{B} -maps.

THEOREM G. This complex K satisfies the "restricted Kan extension condition," in which the omitted face is not allowed to be the first or the last.

This result provides everything we need for composition up to homotopy etc., and allows the formation of the category of $W\mathcal{B}$ -spaces and homotopy classes of homotopy \mathcal{B} -maps.

5. Structure theory. The following theorem is essentially due to Adams.

THEOREM H. Given a $W\mathcal{B}$ -space X , there is a universal monoid MX equipped with a homotopy \mathcal{B} -map $i: X \rightarrow MX$, such that any homotopy

\mathcal{Q} -map $f: X \rightarrow G$ to a monoid G factors uniquely as $g \circ i$ with g a monoid homomorphism. Moreover, if X is a CW-complex the map i is a homotopy equivalence.

Our main technical result for proving Theorem A is:

THEOREM J. *Let X be an E -space, so that in particular it admits a $W\mathcal{Q}$ -structure. Then the classifying space [2] BMX is also an E -space.*

For Theorem A we then define $BX = BMX$.

6. Cohomology theories. Given an E -space Y such that Y is a CW-complex and $\pi_0(Y)$ is a group, we define [1] a graded additive cohomology theory on CW-pairs by setting

$$t^n(X, A) = [X/A, B^n Y], \quad t^{-n}(X, A) = [X/A, \Omega^n Y] \quad \text{for } n \geq 0,$$

whose coefficient groups vanish for $n > 0$. Let us call a cohomology theory with this property *connective*.

THEOREM K. *Every connective graded additive cohomology theory on CW-pairs arises from some such E -space Y , which is uniquely determined up to homotopy equivalence of E -spaces.*

In particular the E -space $\mathbf{Z} \times BU$ gives rise to the connective K -theory cK . This is more usually obtained by appealing to Bott periodicity and killing off the unwanted coefficient groups. In other cases we cannot appeal to Bott periodicity:

DEFINITION. We define *connective piecewise linear K -theory* cK_{PL} by using the E -space $\mathbf{Z} \times BPL$: for $n > 0$ we set

$$cK_{PL}^n(X, A) = [X/A, B^n(\mathbf{Z} \times BPL)].$$

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