

# EXTENDING AUTOMORPHISMS ON PRIMARY GROUPS

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Let  $G$  be a  $p$ -primary abelian group, that is, let  $G$  be a commutative group each element of which has finite order equal to a power of a fixed prime  $p$ . We suppose that  $G$  is written additively and we say that an element  $g \in G$  is divisible by a positive integer  $n$  if there is a solution in  $G$  to the equation  $nx = g$ . Note that 0 is divisible by every positive integer and that the collection of all the elements of  $G$  with this property is a subgroup of  $G$ . This subgroup is denoted by  $p^\omega G$ . It is immediate that  $p^\omega(G/p^\omega G) = 0$ . We can define  $p^\alpha G$  for any ordinal  $\alpha$  inductively by taking intersections at limit ordinals and letting  $p^{\alpha+1}G = p(p^\alpha G)$ .

It has been known for a long time that if the primary group  $G$  is countable, then any automorphism of  $p^\omega G$  can be extended to an automorphism of  $G$ ; this result is implicitly contained in Zippin's proof [7] of Ulm's theorem. It is worth noting that if  $G$  is countable, then  $G/p^\omega G$  is a direct sum of cyclic groups according to a classical result due to Prüfer [6]: if  $G$  is a countable primary group such that  $p^\omega G = 0$ , then  $G$  must be a direct sum of cyclic groups. This suggests the possibility of extending Zippin's result to the case where  $G$  is any primary group with the property that  $G/p^\omega G$  is a direct sum of cyclic groups. Recently, this result has been obtained by Hill and Megibben [3], [4], in a more general setting, and independently by Crawley [1]. What seems to the author as being a more striking generalization is the result being announced in this note, which says that the subgroup does not have to be  $p^\omega G$  in order to extend height-preserving automorphisms.

By the height,  $h_G(x)$ , of an element  $x$  in  $G$  we mean the largest ordinal  $\alpha$  such that  $x \in p^\alpha G$  if such a largest ordinal  $\alpha$  exists. Since  $p^\beta G = \bigcap_{\alpha < \beta} p^\alpha G$  when  $\beta$  is a limit, the only failure of the existence of such a largest ordinal is when  $x \in p^\alpha G$  for all  $\alpha$ . In that case, we put  $h_G(x) = \infty$  and adopt the convention that  $\infty > \alpha$  for all ordinals  $\alpha$ . If  $\pi$  is an automorphism of a subgroup  $H$  of  $G$ , we say that  $\pi$  is height-preserving in  $G$  if  $h_G(x) = h_G(\pi(x))$  for all  $x$  in  $H$ . Denote by  $\mathcal{A}^\alpha(H)$  the group of automorphisms of  $H$  that preserve heights in  $G$  and let  $\mathcal{A}_H(G)$  denote the group of automorphisms of  $G$  that map  $H$  onto  $H$ . Naturally, we write  $\mathcal{A}(G)$  for  $\mathcal{A}^\alpha(G)$  since the latter is the full automorphism group of  $G$ , and we write  $\mathcal{A}(G)$  instead of  $\mathcal{A}_0(G)$  for the

same reason. Observe, more generally, that  $\mathfrak{Q}(G) = \mathfrak{Q}_H(G)$  for any invariant subgroup of  $G$ ; in particular,  $\mathfrak{Q}(G) = \mathfrak{Q}_{p^\omega G}(G)$ . On the other hand, if  $\pi \in \mathfrak{Q}_H(G)$  for any subgroup  $H$  of  $G$ , then the restriction of  $\pi$  to  $H$  is necessarily in  $\mathfrak{Q}^G(H)$ . Thus there is a natural map:  $\mathfrak{Q}_H(G) \rightarrow \mathfrak{Q}^G(H)$ .

**THEOREM.** *Suppose that  $G$  is a primary abelian group and that  $H$  is a subgroup of  $G$  such that  $G/H$  is a direct sum of cyclic groups. Then the natural map  $\mathfrak{Q}_H(G) \rightarrow \mathfrak{Q}^G(H)$  is epic, that is, any height-preserving automorphism of  $H$  can be extended to an automorphism of  $G$ .*

The proof will appear elsewhere [2]. Since  $\mathfrak{Q}_{p^\omega G}(G) = \mathfrak{Q}(G)$  and since  $\mathfrak{Q}^G(p^\omega G) = \mathfrak{Q}(p^\omega G)$ , we have

**COROLLARY (HILL AND MEGIBBEN [3], CRAWLEY [1]).** *If  $G/p^\omega G$  is a direct sum of cyclic groups, then  $\mathfrak{Q}(G) \rightarrow \mathfrak{Q}(p^\omega G)$  is epic.*

The next result is related to [5].

**COROLLARY.** *If  $L$  is a large subgroup of the primary group  $G$ , then the natural map  $\mathfrak{Q}(G) = \mathfrak{Q}_L(G) \rightarrow \mathfrak{Q}^G(L)$  is epic.*

#### REFERENCES

1. P. Crawley, *Abelian  $p$ -groups determined by their Ulm sequences*, Pacific J. Math. **22** (1967), 235–239.
2. P. Hill, *Endomorphism rings generated by units*, (to appear).
3. P. Hill and C. Megibben, *Extending automorphisms and lifting decompositions in abelian groups*, Math. Ann. **175** (1968), 159–168.
4. ———, *Direct sums of countable groups and generalizations*, Proc. of the Montpellier Conference, (to appear).
5. C. Megibben, *Large subgroups and small homomorphisms*, Michigan Math. J. **13** (1966), 153–160.
6. H. Prüfer, *Unendliche abelsche Gruppen von Elementen endlicher Ordnung*, Dissertation, Berlin, 1921.
7. L. Zippin, *Countable torsion groups*, Ann. of Math. **36** (1935), 86–99.

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