Since the factors in the first two sets of brackets are finite Blaschke products and the zero in the third is a convex combination of such, and since the coefficients are nonnegative and sum to 1, the proof is complete.

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EXACTNESS OF INVERSE LIMITS

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I. The problem of this investigation is to characterize those small categories $X$ for which the inverse limit

$$\text{proj lim}_X : AB^X \to AB$$

is exact. Here $AB$ is the category of abelian groups, and $AB^X$ is the category of functors from $X$ to $AB$. In this context I conjecture the following

THEOREM I. Let $X$ be a small category. Then the following assertions are equivalent:

1. The inverse limit $\text{proj lim}_X : AB^X \to AB$ is exact.
2. For every abelian category $\mathcal{A}$ with exact direct products $\mathbf{y}$ the inverse limit $\text{proj lim}_X : \mathcal{A}^X \to \mathcal{A}$ is exact.
3. Every connected component $Y$ of $X$ contains an object $y$ together with an endomorphism $e \in Y(y, y)$ such that the following properties are satisfied:
   i. $y$ is a smallest object of $Y$, i.e., for any object $z \in Y$ there is a morphism $y \to z$.
   ii. $e$ equalizes any two morphisms with the same codomain and domain $y$, i.e., any diagram $y \to z \to y$ is commutative.

At present, I can prove the equivalence of (1) and (2) and the implication (3) $\Rightarrow$ (1) in general, i.e., without any additional condition on $X$. The implication (1) $\Rightarrow$ (3) holds at least if one of the following conditions on $X$ is satisfied:
(a) $X$ is a monoid, i.e., has exactly one object. In this case (3) means that there is a right zero element $e \in X$ with $ae = e$ for all $a \in X$.

(b) $X$ satisfies the condition

$(F2)'$: Every commutative diagram $x_1 \xrightarrow{f} x_2 \xrightarrow{g} x_3$ can be extended to a commutative diagram $x_0 \xrightarrow{h} x_1 \xrightarrow{f} x_2 \xrightarrow{g} x_3$.

This condition is satisfied in particular if the connected components $Y$ of $X$ are filtered from below (see also [1, p. 7]), i.e., if $X$ satisfies the condition

$(F2)'_0$: Any diagram $x_1 \xrightarrow{f} x_2$ in $X$ can be completed to a commutative diagram $x_0 \xrightarrow{h} x_1 \xrightarrow{f} x_2$.

II. The following steps are taken in the proof of Theorem I.

Step 1. It is clear that for any abelian category $\mathfrak{A}$ with exact direct products the inverse limit $\lim_{\mathfrak{A}}: \mathfrak{X} \to \mathfrak{A}$ is exact if and only if for every connected component $Y$ of $X$ the inverse limit $\lim_{\mathfrak{Y}}: \mathfrak{Y} \to \mathfrak{A}$ is exact. Hence one may assume without loss of generality that $X$ itself is connected.

Step 2. Let $\mathbb{Z}$ denote the group of integers, and let $\mathfrak{G}: X \to \mathfrak{A}$ be the constant functor with values $\mathbb{Z}$. Let

$$\text{Hom}_X: (\mathfrak{A}^X)^0 \times \mathfrak{X} \to \mathfrak{A}$$

be the formal Hom-functor, defined for any abelian category $\mathfrak{A}$ with direct products (see e.g. [4, p. 145ff]). In the same fashion let

$$\otimes: \mathfrak{A}^X \times \mathfrak{X}^0 \to \mathfrak{A}$$

be the tensor product, defined for any abelian category $\mathfrak{A}$ with direct sums (loc. cit.). It is easily seen that

$$\lim_{\mathfrak{X}} = \text{Hom}_X(\mathfrak{G},?): \mathfrak{X} \to \mathfrak{A},$$

respectively

$$\text{inj lim}_{\mathfrak{X}} = \mathfrak{G} \otimes?: \mathfrak{X}^0 \to \mathfrak{A}$$

where $X^0$ is the dual category of $X$.

Using these notions we prove the

**Proposition.** Let $X$ be a small category. Then the following properties are equivalent.

1. $\lim_{\mathfrak{X}}: \mathfrak{A}^X \to \mathfrak{A}$ is exact.
2. For any abelian category $\mathfrak{A}$ with exact direct products the functor $\lim_{\mathfrak{X}}: \mathfrak{A}^X \to \mathfrak{A}$ is exact.
3. The object $\mathfrak{G} \in \mathfrak{A}^X$ is projective.
Moreover, if (1)-(3) are satisfied then for any abelian category \( \mathcal{A} \) with exact direct sums the direct limit

\[
\text{inj limit } \mathcal{A}^{X^0} \to \mathcal{A}
\]

is exact.

The basic observation for this proof is that \( \mathcal{Z} \) is projective if and only if it is \( X \)-representable in the sense of Eilenberg-Mac Lane [2], i.e., if and only if the canonical epimorphism

\[
\phi: \prod_{x \in X} ZX(x, ?) \to \mathcal{Z},
\]

\[id_x \mapsto 1 \in \mathcal{Z}x = Z\]
splits. Here \( ZX(x, y) \) is the free abelian group with basis \( X(x, y) \).

The preceding proposition can be generalized to the

**Proposition.** For a small category \( X \) and a nonnegative integer \( n \) the following assertions are equivalent:

1. The \((n+1)^{st}\) right derived functor \((\text{proj limit})^{n+1}_x\) of \( \text{proj limit}_X: AB^X \to AB \) is zero.
2. The \((n+1)^{st}\) right derived functor \((\text{proj limit})^{n+1}_x\) of \( \text{proj limit}_X: \mathcal{A}^X \to \mathcal{A} \) is zero for any abelian category \( \mathcal{A} \) with exact direct products.
3. The projective dimension of \( Z \) is smaller than or equal to \( n \).

Step 3. The most important step in proving Theorem I is the following proposition which is a special case of the implication (1) \( \to \) (3) of Theorem I.

**Proposition.** Let \( X \) be a small, connected category. Assume that \( \text{proj limit}_X: AB^X \to AB \) is exact, and that \( X \) is filtered from below, i.e., satisfies \( (F2)^0 \). Then there are an object \( x \in X \) and an endomorphism \( e \in X(x, x) \) satisfying the conditions (i) and (ii) of (3), Theorem I.

The proof of this proposition starts with the fact that the canonical isomorphism

\[
\phi: \prod_{x \in X} ZX(x, ?) \to \mathcal{Z}
\]
splits.

Step 4. Under the hypotheses (a), respectively (b), on \( X \) the implication (1) \( \rightarrow \) (3) of Theorem I (assume \( X \) connected without loss of generality) is proven in the following way: assume that \( \text{proj limit}_X: AB^X \to AB \) is exact. From the proposition of the second step we conclude that \( \text{inj limit}_X^0: AB^X^0 \to AB \) is exact. Moreover \( X^0 \) satisfies the dual conditions (a) \( ^0 \) respectively (b) \( ^0 \) of (a) respectively (b). Under these
hypotheses it was shown in [3] respectively [5] that \( X^0 \) is filtered from above. Hence \( X \) is filtered from below. Using this fact and the proposition of the third step we obtain that \( X \) satisfies the condition (3) of Theorem I.

Step 5. In order to prove Theorem I in general it is sufficient to prove the following

CONJECTURE: Let \( X \) be a connected, small category and assume that \( \text{inj lim}_X: AB^X \to AB \) is exact. Then \( X \) is filtered from above.

I conjecture this result because it is true if \( X \) is a monoid [3] or satisfies (F2') [5], and no counterexamples are known.

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