Since the factors in the first two sets of brackets are finite Blaschke products and the zero in the third is a convex combination of such, and since the coefficients are nonnegative and sum to 1, the proof is complete.

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EXACTNESS OF INVERSE LIMITS

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I. The problem of this investigation is to characterize those small categories X for which the inverse limit

$$\operatorname{proj}\lim_{X}: AB^X \to AB$$

is exact. Here AB is the category of abelian groups, and AB^X is the category of functors from X to AB. In this context I conjecture the following

THEOREM I. *Let X be a small category. Then the following assertions are equivalent*:

(1) *The inverse limit* $\operatorname{proj}\lim_X$: $AB^X \to AB$ *is exact.*

(2) *For every abelian category* $\mathfrak{A}$ *with exact direct products, the inverse limit* $\operatorname{proj}\lim_X$: $\mathfrak{A}^X \to \mathfrak{A}$ *is exact.*

(3) *Every connected component Y of X contains an object y together with an endomorphism $e \in Y(y, y)$ such that the following properties are satisfied*:

(i) *y is a smallest object of Y, i.e., for any object $z \in Y$ there is a morphism $y \to z$.*

(ii) *e equalizes any two morphisms with the same codomain and domain y, i.e., any diagram $y \xrightarrow{e} y \rightrightarrows z$ is commutative.*

At present, I can prove the equivalence of (1) and (2) and the implication (3)$\rightrightarrows$(1) in general, i.e., without any additional condition on X. The implication (1)$\rightrightarrows$(3) holds at least if one of the following conditions on X is satisfied:

(a) X is a monoid, i.e., has exactly one object. In this case (3) means that there is a right zero element $e \in X$ with $\alpha e = e$ for all $\alpha \in X$.

(b) X satisfies the condition

(F2′)°: Every commutative diagram $x_1 \rightrightarrows x_2 \rightarrow x_3$ can be extended to a commutative diagram $x_0 \rightarrow x_1 \rightrightarrows x_2 \rightarrow x_3$.

This condition is satisfied in particular if the connected components Y of X are filtered from below (see also [**1**, p. 7]), i.e., if X satisfies the condition

(F2)°: Any diagram $x_1 \rightrightarrows x_2$ in X can be completed to a commutative diagram $x_0 \rightarrow x_1 \rightrightarrows x_2$.

II. The following steps are taken in the proof of Theorem I.

Step 1. It is clear that for any abelian category $\mathfrak{A}$ with exact direct products the inverse limit proj $\lim_X: \mathfrak{A}^X \rightarrow \mathfrak{A}$ is exact if and only if for every connected component Y of X the inverse limit proj $\lim_Y: \mathfrak{A}^Y \rightarrow \mathfrak{A}$ is exact. Hence one may assume without loss of generality that X itself is connected.

Step 2. Let $\boldsymbol{Z}$ denote the group of integers, and let $\mathfrak{Z}: X \rightarrow AB$ be the constant functor with values $\boldsymbol{Z}$. Let

$$\mathrm{Hom}_X: (AB^X)^0 \times \mathfrak{A}^X \rightarrow \mathfrak{A}$$

be the formal Hom-functor, defined for any abelian category $\mathfrak{A}$ with direct products (see e.g. [**4**, p. 145ff]). In the same fashion let

$$\underset{X^0}{\otimes} : AB^X \times \mathfrak{A}^{X^0} \rightarrow \mathfrak{A}$$

be the tensor product, defined for any abelian category $\mathfrak{A}$ with direct sums (loc. cit.). It is easily seen that

$$\text{proj}\lim_X = \mathrm{Hom}_X(\mathfrak{Z}, ?): \mathfrak{A}^X \rightarrow \mathfrak{A},$$

respectively

$$\text{inj}\lim_{X^0} = \mathfrak{Z} \underset{X^0}{\otimes} ?: \mathfrak{A}^{X^0} \rightarrow \mathfrak{A}$$

where X^0 is the dual category of X.

Using these notions we prove the

PROPOSITION. *Let X be a small category. Then the following properties are equivalent.*

(1) proj $\lim_X$: *$AB^X \rightarrow AB$ is exact.*

(2) *For any abelian category $\mathfrak{A}$ with exact direct products the functor* proj $\lim_X$: *$\mathfrak{A}^X \rightarrow \mathfrak{A}$ is exact.*

(3) *The object $\mathfrak{Z} \in AB^X$ is projective.*

Moreover, if (1)–(3) *are satisfied then for any abelian category* $\mathfrak{A}$ *with exact direct sums the direct limit*

$$\operatorname{inj}\lim_{X^0}: \mathfrak{A}^{X^0} \to \mathfrak{A}$$

is exact.

The basic observation for this proof is that $\mathfrak{Z}$ is projective if and only if it is X-representable in the sense of Eilenberg-Mac Lane [2], i.e., if and only if the canonical epimorphism

$$\phi: \coprod_{x \in X} \boldsymbol{Z}X(x, ?) \to \mathfrak{Z},$$

$$id_x \rightsquigarrow 1 \in \mathfrak{Z}x = \boldsymbol{Z}$$

splits. Here $\boldsymbol{Z}X(x, y)$ is the free abelian group with basis $X(x, y)$.

The preceding proposition can be generalized to the

PROPOSITION. *For a small category X and a nonnegative integer n the following assertions are equivalent*:

(1) *The* $(n+1)$*st right derived functor* $(\operatorname{proj}\lim_X)_{n+1}$ *of* $\operatorname{proj}\lim_X$: $AB^X \to AB$ *is zero.*

(2) *The* $(n+1)$*st right derived functor* $(\operatorname{proj}\lim_X)_{n+1}$ *of* $\operatorname{proj}\lim_X$: $\mathfrak{A}^X \to \mathfrak{A}$ *is zero for any abelian category* $\mathfrak{A}$ *with exact direct products.*

(3) *The projective dimension of* $\boldsymbol{Z}$ *is smaller than or equal to n.*

Step 3. The most important step in proving Theorem I is the following proposition which is a special case of the implication (1)$\rightrightarrows$(3) of Theorem I.

PROPOSITION. *Let X be a small, connected category. Assume that* $\operatorname{proj}\lim_X: AB^X \to AB$ *is exact, and that X is filtered from below, i.e., satisfies* (F2)°. *Then there are an object* $x \in X$ *and an endomorphism* $e \in X(x, x)$ *satisfying the conditions* (i) *and* (ii) *of* (3), *Theorem* I.

The proof of this proposition starts with the fact that the canonical isomorphism

$$\phi: \coprod_{x \in X} \boldsymbol{Z}X(x, ?) \to \mathfrak{Z}$$

splits.

Step 4. Under the hypotheses (a), respectively (b), on X the implication (1)$\rightrightarrows$(3) of Theorem I (assume X connected without loss of generality) is proven in the following way: assume that $\operatorname{proj}\lim_X: AB^X \to AB$ is exact. From the proposition of the second step we conclude that $\operatorname{inj}\lim_{X^0}: AB^{X^0} \to AB$ is exact. Moreover X^0 satisfies the dual conditions (a)^0 respectively (b)^0 of (a) respectively (b). Under these hypotheses it was shown in [3] respectively [5] that X^0 is filtered from above. Hence X is filtered from below. Using this fact and the proposition of the third step we obtain that X satisfies the condition (3) of Theorem I.

Step 5. In order to prove Theorem I in general it is sufficient to prove the following

CONJECTURE: Let X be a connected, small category and assume that inj $\lim_X$: $AB^X \to AB$ is exact. Then X is filtered from above.

I conjecture this result because it is true if X is a monoid [3] or satisfies (F2′) [5], and no counterexamples are known.

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