ON THE CHARACTERISTIC ROOTS OF TOURNAMENT MATRICES

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A tournament matrix $A=(a_{ij})$ of order n is a matrix of zeros and ones whose main diagonal elements are zeros and all other elements satisfy $a_{ij}+a_{ji}=1$ for $i\neq j$. See, for instance [5].

Such matrices have recently been studied in a large number of papers. But not much seems to be known about their characteristic roots. Since they are nonnegative matrices whose two greatest row-sums are less than or equal to $n-1$ and $n-2$, respectively, it follows from [2] that they lie in the interior or on the boundary of the circle

$$|z| \leqq ((n-1)(n-2))^{1/2}.$$

In this paper, this result will be improved.

THEOREM. *Let A be a tournament matrix of order n with characteristic roots $\omega_1, \omega_2, \cdots, \omega_n$, and $R(\omega_\nu)$ the real part of ω_ν. Assume that*

$$|\omega_1| \geqq |\omega_2| \geqq \cdots \geqq |\omega_n|.$$

Then

$$-\tfrac{1}{2} \leqq R(\omega_\nu) \leqq \tfrac{1}{2}(n-1),$$

and more exactly

$$\omega_1 \leqq \tfrac{1}{2}(n-1) \quad \textit{and} \quad |\omega_\nu| \leqq \left(\frac{n(n-1)}{2\nu}\right)^{1/2} \quad \textit{for} \quad \nu \geqq 2.$$

PROOF. Let B be the symmetric matrix $\frac{1}{2}(A+A')$. All its main diagonal elements are zeros and all other elements equal $\frac{1}{2}$. Since B is a generalized stochastic matrix with row-sum $\frac{1}{2}(n-1)$, its greatest root is $\frac{1}{2}(n-1)$. Moreover, it follows from [3] that the nontrivial roots remain unchanged if we subtract from all the elements of each column the number $\frac{1}{2}$. We obtain the diagonal matrix $D(-\frac{1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2})$. Hence B has the root $\frac{1}{2}(n-1)$ and $n-1$ roots $-\frac{1}{2}$.

In 1902, I. Bendixson [1] proved the following theorem.

Let T be a matrix with real elements, S the symmetric matrix $\frac{1}{2}(T+T')$, and M and m the maximum and the minimum of the char-

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acteristic roots of S, respectively, then the real part of any root η_ν of T satisfies

$$m \leqq R(\eta_\nu) \leqq M.$$

Applying this theorem to the matrix A it follows that

$$-\tfrac{1}{2} \leqq R(\omega_\nu) \leqq \tfrac{1}{2}(n-1). \tag{1}$$

Since A is nonnegative, it follows from the theorem of Frobenius (see [4]) that ω_1 is positive, hence by (1)

$$0 < \omega_1 \leqq \tfrac{1}{2}(n-1), \ |\omega_\nu| \leqq \tfrac{1}{2}(n-1) \qquad \text{for } \nu \geqq 2,$$

and it follows from (1) that

$$-\tfrac{1}{2} \leqq R(\omega_\nu) \leqq |\omega_\nu| \leqq \tfrac{1}{2}(n-1). \tag{2}$$

It can easily be seen that this result cannot be improved in general. Let n be an odd integer and let us assume that each of the n players wins $\frac{1}{2}(n-1)$ games. Then the tournament matrix is a generalized stochastic matrix with row-sum $\frac{1}{2}(n-1)$, hence $\omega_1 = \frac{1}{2}(n-1)$. Since the trace of A is zero, the sum of the real parts of all the roots different from ω_1 must be $-\frac{1}{2}(n-1)$, and by (2) $R(\omega_\nu) = -\frac{1}{2}$ for $\nu = 2, 3, \cdots, n$.

For $\nu \geqq 2$ the inequality (2) can be improved.

It follows from a theorem of I. Schur [6] that

$$|\omega_1|^2 + |\omega_2|^2 + \cdots + |\omega_n|^2 \leqq \tfrac{1}{2} n(n-1).$$

Hence

$$\nu|\omega_\nu|^2 \leqq |\omega_1|^2 + |\omega_2|^2 + \cdots + |\omega_\nu|^2 \leqq \tfrac{1}{2} n(n-1),$$

$$|\omega_\nu| \leqq \left(\frac{n(n-1)}{2\nu}\right)^{1/2}.$$

It follows in particular that if A has imaginary roots, then

$$|I(\omega_\nu)| \leqq \left(\frac{n(n-1)}{6}\right)^{1/2}.$$

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A GENERAL MEAN VALUE THEOREM[1]

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We present here in general terms the idea of the mean of a function relative to a "weight function" $w(\xi, \nu)$, special instances and applications appearing elsewhere [1], [2].

1. **The weight function.** If $X=[h, k]$ is a real interval, (I, A, μ) a finite measure space with $\mu(I)=1$, and $w(\xi, \nu)$ a nonnegative function on $X\times I$ which, for each ν of I, is measurable, and positive a.e. on X, then the indefinite integral

$$W(x, \nu) = \int_h^x w(\xi, \nu)d\xi \tag{1}$$

is defined on $X\times I$, and the function

$$\mathcal{W}(x) = \int_I W(x, \nu)d\mu, \qquad x\in X$$

which we assume to exist, is continuous and strictly increasing on X, as is $W(x, \nu)$ for each ν.

2. **The mean of a function.** Let $x(\nu)$ be any μ-integrable function on I to X for which the integral functional

$$\mathcal{W}_x = \int_I W(x(\nu), \nu)d\mu$$

exists. Let x_u be the *essential* upper bound of $x(\nu)$ on I, i.e., the g.l.b. of all real x for which $\mu\{\nu \mid x(\nu)>x\}=0$, the essential lower bound

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