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A GENERAL MEAN VALUE THEOREM[1]

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We present here in general terms the idea of the mean of a function relative to a "weight function" $w(\xi, \nu)$, special instances and applications appearing elsewhere [1], [2].

1. **The weight function.** If $X=[h, k]$ is a real interval, (I, A, μ) a finite measure space with $\mu(I)=1$, and $w(\xi, \nu)$ a nonnegative function on $X\times I$ which, for each ν of I, is measurable, and positive a.e. on X, then the indefinite integral

$$W(x, \nu) = \int_h^x w(\xi, \nu)d\xi \tag{1}$$

is defined on $X\times I$, and the function

$$\mathcal{W}(x) = \int_I W(x, \nu)d\mu, \qquad x \in X$$

which we assume to exist, is continuous and strictly increasing on X, as is $W(x, \nu)$ for each ν.

2. **The mean of a function.** Let $x(\nu)$ be any μ-integrable function on I to X for which the integral functional

$$\mathcal{W}_x = \int_I W(x(\nu), \nu)d\mu$$

exists. Let x_u be the *essential* upper bound of $x(\nu)$ on I, i.e., the g.l.b. of all real x for which $\mu\{\nu \mid x(\nu)>x\}=0$, the essential lower bound

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x_l being analogously defined. Clearly $x(\nu)$ is constant μ-a.e. if and only if $x_l = x_u$.

Referring to (1), it is apparent that the continuous, strictly increasing function

$$B(x) = \int_I \int_{x(\nu)}^{x} w(\xi, \nu) d\xi d\mu = \mathcal{W}(x) - \mathcal{W}_x$$

has a unique zero b on X, namely

$$b = \mathcal{W}^{-1}(\mathcal{W}_x)$$

called *the mean of* $x(\nu)$ *relative to* $w(\xi, \nu)$. For, if $x(\nu)$ is a constant x_0, μ-a.e., we have $B(x_0) = 0$; otherwise we see that $B(x_l) < 0 < B(x_u)$, so that $B(b) = 0$ for a unique b on (x_l, x_u).

3. The principal theorem. For an arbitrary bounded, monotone nondecreasing function $g(\xi)$ on X, we analogously define

$$G(x, \nu) = \int_h^x g(\xi) w(\xi, \nu) d\xi$$

on $X \times I$, and assume the existence of

$$\mathcal{G}(x) = \int_I G(x, \nu) d\mu, \qquad x \in X$$

and of

$$\mathcal{G}_x = \int_I G(x(\nu), \nu) d\mu.$$

For the function

$$C(x) = \int_I \int_{x(\nu)}^{x} g(\xi) w(\xi, \nu) d\xi d\mu = \mathcal{G}(x) - \mathcal{G}_x$$

we then have the basic

THEOREM.

$$C(b) \leqq 0 \tag{2}$$

or

$$\mathcal{G}(\mathcal{W}^{-1}(\mathcal{W}_x)) \leqq \mathcal{G}_x.$$

Equality holds if and only if $x(\nu) \equiv b$, μ*-a.e., or* $g(\xi) \equiv g(b)$ *everywhere on the open interval* (x_l, x_u).

The inequality is rendered transparent by splitting I into the μ-measurable subsets L, Z, U on which $x(\nu) \lesseqgtr b$, respectively, and observing that

$$\begin{aligned} -C(b) &= g(b)B(b) - C(b) \\ &= \int_L \int_{x(\nu)}^b \{g(b) - g(\xi)\} w(\xi, \nu) d\xi d\mu \\ &\quad + \int_U \int_b^{x(\nu)} \{g(\xi) - g(b)\} w(\xi, \nu) d\xi d\mu \geqq 0. \end{aligned}$$

4. **Two applications.** In the simplest case, $w(\xi, \nu) \equiv 1$, (2) is Jensen's inequality

$$\mathcal{G}\left(\int_I x(\nu) d\mu\right) \leqq \int_I \mathcal{G}(x(\nu)) d\mu$$

for the general convex function $\mathcal{G}(x) = G(x) = G(x, \nu) = \int_h^x g(\xi) d\xi$ [**3**, §13.34, §18.43]. A particular instance is mentioned in [**2**, §3].

Again, if we take $h > 0$ and set $w(\xi, \nu) = \xi^{s-1}$, s real, we find that b is the "mean of order s" of $x(\nu)$:

$$M_s = \left\{\left(\int_I x^s(\nu) d\mu\right)^{1/s}, \ s \neq 0; \exp \int_I \log x(\nu) d\mu, \ s = 0\right\},$$

and (2) yields the classical inequality

$$M_s \leqq M_t \qquad \text{for } s < t$$

if one takes $g(\xi) = \xi^{t-s}$ [**1**], [**2**].

These are trivial examples of the "separable" case $w(\xi, \nu) = w(\xi) f(\nu)$. Nonseparable cases arise naturally in physical problems, as indicated below.

5. **A "minimax" principle.** Let $y(\nu)$ be a second function such as $x(\nu)$, and suppose that

$$\text{(3)} \qquad \int_I \int_{x(\nu)}^{y(\nu)} w(\xi, \nu) d\xi d\mu = 0.$$

This is equivalent to the assertion that $y(\nu)$ and $x(\nu)$ have the same mean b relative to $w(\xi, \nu)$, and it follows at once from the Theorem (applied to $y(\nu)$) that

$$
\begin{aligned}
\int_I \int_{x(\nu)}^{y(\nu)} g(\xi) w(\xi, \nu) d\xi d\mu &\equiv \int_I \int_{x(\nu)}^{b} - \int_I \int_{y(\nu)}^{b} \\
&\geqq \int_I \int_{x(\nu)}^{b} g(\xi) w(\xi, \nu) d\xi d\mu.
\end{aligned} \tag{4}
$$

If we regard $x(\nu)$ as initial temperature distribution on an interval I, of mass $m(\nu)$ per unit length ($d\mu = m(\nu)d\nu$), and specific heat $w(\xi, \nu)$, then (3) singles out the energy conserving distributions $y(\nu)$, and (4) (with $g(\xi) = -1/\xi$) shows that, among these, the entropy change is greatest for the uniform mean temperature $y(\nu) \equiv b$.

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