flat manifold is necessarily flat. This answers a conjecture of Auslander and Wolf posed in [5].

(F) The notion of totally convex sets can be used to study isometries of complete manifolds of nonnegative curvature.

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ON MIXING IN INFINITE MEASURE SPACES

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1. E. Hopf [6, p. 66] suggested that strong mixing in infinite measure spaces should be defined by a limit statement on certain ratios; Krickeberg [9] made this precise in the context of topological measure spaces. In this paper we shall consider a different concept of strong mixing, meaningful also without existence of a topological structure. Our notion coincides with the usual concept of strong mixing in the case of finite measure spaces and seems to be the proper generalization to the infinite measure case in that it is exactly the concept needed to carry over certain theorems on mixing which hold in finite measure spaces.

Given a sequence (A_n) of measurable sets on a measure space $(\Omega, \mathcal{A}, \mu)$, the intersection $\mathcal{R}(A_n)$ of the σ-algebras $\mathcal{B}_k(A_n)$ generated by $A_k, A_{k+1}, \cdots$ will be called the *remote σ-algebra* of (A_n). A sequence (A_n) is called *remotely trivial*, iff $\mathcal{R}(A_n)$ is *trivial*, i.e., contains only

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null sets and their complements. The sequence (A_n) is called *semiremotely trivial* iff every subsequence contains a further subsequence which is remotely trivial. It is known [12] that a sequence of sets in a finite measure space is mixing if and only if it is semiremotely trivial; hence a measure-preserving transformation T in a finite measure space is mixing if and only if for any measurable A, the sequence $A, T^{-1}A, T^{-2}A, \cdots$ is semiremotely trivial. We call a measure-preserving transformation T in a σ-finite measure space *mixing* iff for every set A with *finite measure*, the sequence $A, T^{-1}A, T^{-2}A, \cdots$ is semiremotely trivial; we call T *completely mixing*, iff this is true for *every* measurable set A. We prove that a measure-preserving transformation T is mixing if and only if $\mu(T^{-n}A\cap A)\to 0$ for every set A of finite measure. Exact endomorphisms are completely mixing, but Kolmogorov automorphisms are only mixing. In fact, complete mixing for arbitrary nonsingular *invertible* transformations is equivalent with the existence of a *finite* equivalent invariant measure for which T is mixing. We also show the equivalence of complete mixing with a mixing condition involving differences of measures (condition (γ) in Theorem 5), introduced by Mrs. Dowker, thus negatively answering her question, whether in infinite measure spaces Kolmogorov automorphisms are mixing in her sense. In infinite measure spaces, however, Kolmogorov automorphisms seem of less interest than *remotely infinite* automorphisms, introduced below. Using the concept of remotely infinite automorphism we solve, within the class of conservative automorphisms, an isomorphy problem asked by Halmos [5].

We further prove for transformations in infinite measure spaces the analogue of a theorem of Blum and Hanson [2]: The mean ergodic theorem in L_p $(p>1)$ holds for all sequences (T^{k_n}) if and only if T is mixing. Finally, we relate complete mixing to unaveraged L_1 convergence to zero.

2. All sets and functions introduced below are assumed measurable; all relations are assumed to hold modulo sets of μ-measure zero. A sequence (A_n) of sets is called *remotely infinite* iff $\mathcal{R}(A_n)$ contains besides the empty set $\emptyset$ only sets of infinite measure; (A_n) is called *semiremotely infinite* iff every subsequence of (A_n) contains a remotely infinite subsequence. The collection of sets of finite measure is denoted by $\mathfrak{F}$.

THEOREM 1. *Assume* $\mu(\Omega)=\infty$. *The following conditions are equivalent for a sequence of sets* A_n *with* $\mu(A_n)$ *bounded by a constant* c:

For each $F\in\mathfrak{F}$, $\mu(A_n\cap F)\to 0$; *for every integer* k, $\lim_n\mu(A_n\cap A_k)=0$; (A_n) *is semiremotely trivial*; (A_n) *is semiremotely infinite.*

If one, and hence all, of the conditions of Theorem 1 hold, the sequence (A_n) is called *mixing.*

COROLLARY 1. *Let* $\mathfrak{A}_0, \mathfrak{A}_1, \cdots$ *be a decreasing sequence of* σ*-algebras contained in* $\mathfrak{A}$ *and let* $\mathfrak{A}_\infty = \bigcap_{i=0}^\infty \mathfrak{A}_i$. *A necessary and sufficient condition that* $\mathfrak{A}_\infty$ *contain only the empty set* $\varnothing$ *and sets of infinite measure, is that for each* $c > 0$ *and for each* $F \in \mathfrak{F}$, $\mu(A_n \cap F) \to 0$ *uniformly in the class of all sequences* (A_n) *with* $A_n \in \mathfrak{A}_n$ *and* $\mu(A_n) < c$.

THEOREM 2. *Let* (A_n) *be a sequence of measurable sets in a* σ*-finite measure space* $(\Omega, \mathfrak{A}, \mu)$. *Then the following conditions* (i) *and* (ii) *are equivalent*:

(i) (A_n) *is semiremotely trivial.*

(ii) $\int_{A_n} f d\mu \to 0$ *for all* $f \in L_1(\Omega, \mathfrak{A}, \mu)$ *with* $\int f d\mu = 0$.

A measurable transformation T of a measure space $(\Omega, \mathfrak{A}, \mu)$ is called *null-preserving* iff $\mu(T^{-1}A) = 0$ holds for all $A \in \mathfrak{A}$ with $\mu(A) = 0$; it is called *measure-preserving* iff $\mu(A) = \mu(T^{-1}A)$ for each $A \in \mathfrak{A}$. Measure-preserving transformations will be called *endomorphisms* for brevity. An endomorphism T on $(\Omega, \mathfrak{A}, \mu)$ is called an *automorphism* if T^{-1} is also an endomorphism on $(\Omega, \mathfrak{A}, \mu)$. A null-preserving transformation T is called *mixing* iff for every set $A \in \mathfrak{F}$, the sequence $T^{-n}A$ is semiremotely trivial; T is called *completely mixing* iff the same is true for *all* $A \in \mathfrak{A}$. By Theorem 2, in the finite measure case, when both notions coincide, they agree with the usual notion of (strong) mixing. Henceforth, we shall assume that $\mu(\Omega) = \infty$.

An endomorphism T is by Theorem 1 mixing iff $f \circ T^n$ tends weakly to zero in L_2 for every $f \in L_2(\Omega, \mathfrak{A}, \mu)$; L_2 may be replaced by any other L_p with $1 < p < \infty$. By Theorem 1 an endomorphism T is mixing iff $\mu(T^{-n}A \cap A) \to 0$, $A \in F$. Endomorphisms T with this property are called of *zero type*; endomorphisms T for which lim sup $\mu(T^{-n}A \cap A) > 0$ if $0 < \mu(A) < \infty$, are called of *positive type.* Hajian and Kakutani [4] asserted the existence of automorphisms of both types. A set $A \in \mathfrak{A}$ is called *invariant* iff $T^{-1}A = A$.

THEOREM 3. *If* T *is an endomorphism of a* σ*-finite measure space* $(\Omega, \mathfrak{A}, \mu)$, *then* Ω *uniquely decomposes into two invariant sets* Ω_0 *and* Ω_+, *such that* T *restricted to* Ω_0 *is of zero type* (*equivalently*: *mixing*) *and* T *restricted to* Ω_+ *is of positive type.*

An endomorphism T of $(\Omega, \mathfrak{A}, \mu)$ is called *exact* iff $\bigcap_{i=0}^\infty T^{-i}\mathfrak{A} = \{\varnothing, \Omega\}$. Examples of exact endomorphisms are provided by unilateral shifts on stochastic processes with infinite invariant measure and trivial remote σ-algebra; e.g., null-recurrent aperiodic Markov chains or aperiodic random walks (see [1] and [8]). Clearly exact endomorphisms are completely mixing. We relate the concept of mixing to the spectral structure of an endomorphism T. T induces an isometric operator U_T in $L_2(\Omega, \mathfrak{A}, \mu)$ by the relation $U_T f = f \circ T, f \in L_2$. T is said to have *Lebesgue spectrum* iff there exist two disjoint index sets I_1 and I_2 (one of which may be empty) and an orthonormal basis of L_2:

$$\{f_{i,k}; i \in I_1, k = 0, \pm 1, \pm 2, \cdots\} \cup \{f_{j,k}; j \in I_2, k = 0, 1, 2, \cdots\},$$

such that $U_T f_{i,k} = f_{i,k+1}$ for all i, k.

THEOREM 4. *If an endomorphism T in an infinite measure space $(\Omega, \mathfrak{A}, \mu)$ has Lebesgue spectrum, then T is mixing.*

In fact the spectral structure and mixing do not depend on the triviality of the remote σ-algebra; it suffices that this σ-algebra contain no nontrivial sets of finite measure. An endomorphism T in an infinite measure space $(\Omega, \mathfrak{A}, \mu)$ is called *remotely infinite* iff $\mathfrak{F} \cap \bigcap_{k=0}^{\infty} T^{-k}\mathfrak{A} = \{\emptyset\}$. An automorphism on $(\Omega, \mathfrak{A}, \mu)$ is called a *Kolmogorov automorphism* iff there exists a σ-algebra $\mathfrak{A}_0$ such that μ restricted to $\mathfrak{A}_0$ is σ-finite and

(1) $T^{-1}\mathfrak{A}_0 \subset \mathfrak{A}_0$;
(2) $\bigcup_{k=0}^{\infty} T^k \mathfrak{A}_0$ generates $\mathfrak{A}$;
(3) $\bigcap_{k=0}^{\infty} T^{-k}\mathfrak{A}_0 = \{\emptyset, \Omega\}$.

If in this definition (3) is replaced by

(4) $\mathfrak{F} \cap \bigcap_{k=0}^{\infty} T^{-k}\mathfrak{A}_0 = \{\emptyset\}$,

T is called a *remotely infinite* automorphism. Remotely infinite automorphisms and endomorphisms have Lebesgue spectrum; therefore remotely infinite automorphisms are mixing. For automorphisms the index set I_2 in the definition of the Lebesgue spectrum is empty. Under some assumptions on the measure space, the cardinality of I_1 is denumerably infinite; this is so if $(\Omega, \mathfrak{A}, \mu)$ is a *Lebesgue space* (see [11]). Thus any two remotely infinite automorphisms in a Lebesgue space are spectrally isomorphic. Halmos [5] asked whether in the class of automorphisms of σ-finite measure spaces, ergodicity was a spectral invariant. Parry [10] has given a negative answer to this problem, constructing a nonergodic dissipative automorphism which is spectrally isomorphic to an ergodic conservative one. Since dissipative automorphisms, at least in Ergodic Theory, are in a sense pathological, and the isomorphy problem can easily be solved for the class of all dissipative automorphisms (see [7]), it seems of interest that ergodicity is not even a spectral invariant in the narrower class of conservative automorphisms: It follows from a result of Blackwell and Freedman [1, Corollary 2] that bilateral shifts on null-recurrent Markov chains are remotely infinite; hence any two such shifts have countable Lebesgue spectrum and therefore are spectrally isomorphic. Let T be the bilateral shift on a null-recurrent irreducible Markov chain of period $d \geqq 2$. Let $T' = T^d$; then T and T' are spectrally isomorphic because a power of an automorphism with countable Lebesgue spectrum has countable Lebesgue spectrum, but T is ergodic and T' is not.

An *invertible transformation* T of Ω is a one-to-one map $T: \Omega \to \Omega$ such that $T^{-1}\mathfrak{A} = \mathfrak{A} = T\mathfrak{A}$. T is called *nonsingular* iff $\mu(A) = 0$ implies $\mu(T^{-1}A) = \mu(TA) = 0$.

THEOREM 5. *Let T be an invertible nonsingular transformation on a σ-finite measure space $(\Omega, \mathfrak{A}, \mu)$. The following conditions are equivalent*:

(α) *T is completely mixing.*

(β) *There is an equivalent invariant probability measure π_0 such that T is mixing on $(\Omega, \mathfrak{A}, \pi_0)$.*

(γ) *For any two probability measures π_1, π_2 equivalent with μ,*

$$\pi_1(T^{-n}A) - \pi_2(T^{-n}A) \to 0 \qquad A \in \mathfrak{A}.$$

THEOREM 6. *Let T be a endomorphism in a σ-finite infinite measure space and let p be a real number with $1 < p < \infty$. If T is mixing and $f \in L_p$, then*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} f \cdot T^{k_i} \right\|_p \to 0 \tag{5}$$

uniformly in the set $\boldsymbol{K}$ of all strictly increasing sequences (k_n) of natural numbers. Conversely, if for every $f \in L_p$ and every $(k_n) \in \boldsymbol{K}$, (5) *holds, then T is mixing.*

The uniform convergence in $\boldsymbol{K}$ may also be shown in the Blum-Hanson theorem (finite measure case).

We let $T^n f = d(\pi \circ T^{-n})/d\mu$ where $f = d\pi/d\mu$.

THEOREM 7. *Let T be a null-preserving point-transformation in a σ-finite measure space $(\Omega, \mathfrak{A}, \mu)$ and assume (α): There exists no invariant μ-continuous probability measure π. Then T is completely mixing if and only if*

$$\lim_n \left\| T^n f \right\|_1 = 0 \tag{6}$$

for all functions $f \in L_1$ with $\int f\, d\mu = 0$. If T is conservative, then the assumption $\pi \ll \mu$ in (α) may be replaced by $\pi \sim \mu$.

As applied to null-recurrent irreducible aperiodic Markov chains, Theorem 7 is a strengthening of a theorem of Orey (cf. [1]), which assumes that f is one-dimensional (defined on integers). Theorem 7 may be also applied to transient Markov chains (cf. [8]).

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