COHOMOLOGY OF CERTAIN STEINBERG GROUPS

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In [3] Steinberg considers the relations satisfied by generators of the Chevalley groups and defines certain abstract groups $\Delta$ and $\Gamma$ via presentation. Let $\Sigma$ be a root system of a simple complex Lie algebra $\mathfrak{g}_C$ and let $K$ be a field of characteristic $p \geq 0$. We consider a set of generators $x_r(t)$ ($r \in \Sigma$, $t \in K$) and the relations

(A) $x_r(t)x_r(u) = x_r(t + u)$ ($r \in \Sigma$; $t, u \in K$),

(B) $x_r(t)x_r(u)x_r(t)^{-1} = x_s(u) \prod x_{ir+js}(C_{ij}; t^iu^j)$

where $(r, s \in \Sigma, r + s \neq 0; t, u \in K)$.

The product in (B) is over all integers $i, j \geq 1$ for which $ir+js \in \Sigma$, taken in lexicographic order. The $C_{ij}$ are certain integers depending only on the structure of $\mathfrak{g}_C$ (cf. [1]). Steinberg defines $w_r(t) = x_r(t)x_r(-t^{-1})x_r(t)$ and $h_r(t) = w_r(t)w_r(-1)$ ($r \in \Sigma$; $t \in K^*$) and considers also the relations

(B') $w_r(t)x_r(u)w_r(t) = x_r(t)(-t^2u)$ ($r \in \Sigma$; $t \in K^*$, $u \in K$),

(C) $h_r(t)h_r(u) = h_r(tu)$ ($r \in \Sigma$; $t, u \in K^*$).

The Steinberg group $\Delta$ is the abstract group generated by the symbols $x_r(t)$ ($r \in \Sigma$; $t \in K$) subject to the relations (A) and (B) if the rank of $\Sigma$ is $> 1$, to the relations (A) and (B') if the rank of $\Sigma = 1$. The Steinberg group $\Gamma$ is the abstract group with the same generators as $\Delta$ subject to the relations of $\Delta$ and in addition subject to the relations (C).

In [1] Chevalley constructs a corresponding Lie algebra $\mathfrak{g}$ over the field $K$ and there is a natural action of the Steinberg groups on $\mathfrak{g}$. One is then led to consideration of the cohomology $H^*(\Delta, \mathfrak{g})$ and $H^*(\Gamma, \mathfrak{g})$.

The author has developed a technique for computation of such cohomology (cf. [2]). This is applied successfully to obtain the following results. Proofs will appear elsewhere.

We denote $\mathfrak{D}(K)$ the module of derivations of $K$. In the case of characteristic $p = 2$ we denote $\mathfrak{L}(K)$ the $K$-linear transformations $L$ of $K$ such that $L(1) = 0$. Since $p = 2$, $\mathfrak{D}(K) \subset \mathfrak{L}(K)$, but in general $\mathfrak{D}(K) \neq \mathfrak{L}(K)$.

**Theorem 1.** $H^1(\Delta, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}) \cong \mathfrak{D}(K)$ in the following cases:

(i) type $A_1$, $p \neq 2$ and $K \neq F_3$;
Theorem 2. $H^1(\Delta, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}) \cong K \oplus D(K)$ in the following cases:

(i) type $A_n$ ($n \geq 2$), $p \mid n + 1$;
(ii) type $D_n$ ($n \geq 4$), $p \neq 2$;
(iii) type $E_6$, $p \neq 3$;
(iv) type $E_7$, $p \neq 2$;
(v) type $E_8$;
(vi) type $F_4$.

Theorem 3. $H^1(\Delta, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}) \cong K \oplus K \oplus D(K)$ in the case: type $D_n$ ($n \geq 4$, $n$ even), $p = 2$.

Theorem 4. $H^1(\Delta, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}) \cong K$ in the case: type $A_1$, $K = F_4$.

Theorem 5. $H^1(\Delta, \mathfrak{g}) \cong K \oplus L(K)$, $H^1(\Gamma, \mathfrak{g}) \cong K \oplus D(K)$ in the case: type $A_1$, $p = 2$.

References


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