

AN ALGEBRA OF SINGULAR INTEGRAL OPERATORS WITH TWO SYMBOL HOMOMORPHISMS

BY H. O. CORDES¹

Communicated by M. H. Protter, May 28, 1968

1. Let $\mathcal{R}_+^{n+1} = \{(x, y) = (x_1, \dots, x_n, y) : x_j, y \in \mathcal{R}, y \geq 0\}$ and let Δ_d, Δ_n denote the two unbounded positive self-adjoint operators of the Hilbert-space $\mathfrak{S} = \mathcal{L}^2(\mathcal{R}_+^{n+1})$ generated by closing the Laplace operator in $C_0^\infty(\mathcal{R}_+^{n+1})$ under Dirichlet and Neumann boundary conditions at $y=0$, respectively.

We propose to study the "convolution algebra" $\mathfrak{A}^\#$ generated by the generalized Riesz-Hilbert-operators

$$(1) \quad \begin{aligned} \Lambda_d &= (1 - \Delta_d)^{-1/2}, \quad \Lambda_n = (1 - \Delta_n)^{-1/2}, \quad S_d = -i\partial/\partial y \Lambda_d, \\ S_n &= -i\partial/\partial y \Lambda_n, \quad S_{d,j} = -i\partial/\partial x_j \Lambda_d, \quad S_{n,j} = -i\partial/\partial x_j \Lambda_n, \\ & \qquad \qquad \qquad j = 1, \dots, n \end{aligned}$$

and later on also will adjoin certain multiplications by continuous functions, to obtain an algebra \mathfrak{A} of singular integral operators on the half-space \mathcal{R}_+^{n+1} .

Both C^* -algebras $\mathfrak{A}^\#$ and \mathfrak{A} have noncompact commutators, but each is commutative modulo a certain larger ideal ($\mathfrak{E}^\#$ and \mathfrak{E} , respectively). We therefore obtain a first symbol function σ_A for $A \in \mathfrak{A}^\#$ (or \mathfrak{A}) which is a continuous complex-valued function over the maximal ideal space of $\mathfrak{A}^\#/\mathfrak{E}^\#$ (or $\mathfrak{A}/\mathfrak{E}$). If σ_A does not vanish, we can invert the operator mod $\mathfrak{E}^\#$ (or \mathfrak{E}), or reduce the singular integral equation $A_n u = f$ to an equation $(1+E)u = g$ with $E \in \mathfrak{E}^\#$ (or \mathfrak{E}).

Now, we find that the ideals $\mathfrak{E}^\#$ and \mathfrak{E} are isomorphic to topological tensor products of the form $\mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{E}^\#, \mathfrak{E} = \mathfrak{C}(\mathfrak{h}) \hat{\otimes} \mathfrak{E}$, with respect to a suitable direct decomposition

$$\mathfrak{S} = \mathfrak{h} \otimes \mathfrak{k}, \quad \mathfrak{h} = \mathcal{L}^2(\mathcal{R}^+), \quad \mathfrak{k} = \mathcal{L}^2(\mathcal{R}^n),$$

where $\mathfrak{C}(\mathfrak{h})$ denotes the compact ideal of \mathfrak{h} , while $\mathfrak{E}^\#$ and \mathfrak{E} are certain algebras of singular integral operators over the boundary \mathcal{R}^{n+1} .

Therefore to each operator $E \in \mathfrak{E}^\#$ (or \mathfrak{E}) there can be associated an operator valued symbol $\tau_E(m) \in \mathfrak{C}(\mathfrak{h})$ such that $1+E$ is Fredholm if and only if $1+\tau_E(m)$ is regular for all m . The construction of a Fredholm inverse for $A \in \mathfrak{A}$ will therefore depend on two symbols: first we invert the operator modulo \mathfrak{E} , if the complex-valued symbol

¹ Supported by contract AF-AFOSR 553-64.

σ_A does not vanish; then we invert an operator $1 + E \pmod{\mathfrak{C}}$, which depends on another, operator-valued symbol.

2. It is well known that the operators (1) have representations as (regular or singular) integral operators. Specifically

$$\Lambda_d = \Lambda_- - \Lambda_+, \quad \Lambda_n = \Lambda_- + \Lambda_+$$

with

$$(2) \quad \Lambda_{\pm}u = (2/\pi)^{1/2}(2\pi)^{-(n+1)/2} \int_{\mathbb{R}_+^{n+1}} K_{n/2}(t_{\pm})t_{\pm}^{-n/2} u(x', y') dx' dy'$$

and

$$(3) \quad t_{\pm} = (|x - x'|^2 + |y \pm y'|^2)^{1/2},$$

where $K_\nu(s)$ denotes the modified Bessel function as in Magnus-Oberhettinger [6, p. 28]. All other operators (1) experience similar decompositions and we therefore may generate \mathfrak{A}^\sharp by the following operators as well, which are integral operators:

$$(4) \quad \Lambda_{\pm}, S_{\pm} = -i\partial/\partial y \Lambda_{\pm}, \quad S_{j,\pm} = -i\partial/\partial x_j \Lambda_{\pm}, \quad j = 1, \dots, n.$$

Note that

$$(5) \quad S_{\pm}u = i(2/\pi)^{1/2}(2\pi)^{-(n+1)/2} \cdot \int_{\mathbb{R}_+^{n+1}} K_{n/2+1}(t_{\pm})(y \pm y')/t_{\pm}^{n/2+1} u(x', y') dx' dy'$$

and

$$(6) \quad S_{j,\pm}u = i(2/\pi)^{1/2}(2\pi)^{-(n+1)/2} \cdot \int_{\mathbb{R}_+^{n+1}} K_{n/2+1}(t_{\pm})(x_j - x'_j)/t_{\pm}^{n/2+1} u(x', y') dx' dy'.$$

3. Let F denote the unitary operator of \mathfrak{S} induced by the Fourier transform, with respect to the x -variable only:

$$(7) \quad Fu(x, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(\xi, y) d\xi \quad \text{for } u \in C_0^\infty(\mathbb{R}_+^{n+1})$$

and let the unitary operator T be defined by

$$(8) \quad (Tu)(x, y) = \sigma^{-1/2}u(x, y/\sigma), \quad (T^{-1}u)(x, y) = \sigma^{1/2}u(x, y\sigma)$$

with $\sigma = (1 + |x|^2)^{1/2}$. Let $U = TF$; then we find that

$$(9) \quad US_{\pm}U^{-1} = P_{\pm}, \quad US_{j,\pm}U^{-1} = x_j/\sigma Q_{\pm}, \quad U\Lambda_{\pm}U^{-1} = 1/\sigma Q_{\pm}$$

with

$$(10) \quad P_{\pm}u = i/\pi \int_0^{\infty} K_1(|y \pm y'|) \operatorname{sgn}(y \pm y')u(x, y')dy'$$

and

$$(11) \quad Q_{\pm}u = 1/\pi \int_0^{\infty} K_0(|y \pm y'|)u(x, y')dy',$$

with $\operatorname{sgn} t = 0, \pm 1$ if $t = 0, > 0, < 0$, resp.

Let $\mathfrak{h} = \mathfrak{L}^2(\mathbb{R}_+^1)$ and, for a moment, let D_a and D_n denote the operators Δ_a and Δ_n as introduced initially, but for $n = 0$. Then we see at once that we may reinterpret P_{\pm}, Q_{\pm} above as operators on \mathfrak{h} and that then

$$(12) \quad \begin{aligned} (1 - D_a)^{-1/2} &= Q_- - Q_+, & (1 - D_n)^{-1/2} &= Q_- + Q_+, \\ -i\partial/\partial y(1 - D_a)^{-1/2} &= P_- - P_+, & -i\partial/\partial y(1 - D_n)^{-1/2} &= P_- + P_+, \end{aligned}$$

while we get

$$(13) \quad \mathfrak{F} = \mathfrak{f} \hat{\otimes} \mathfrak{h}, \quad \mathfrak{f} = \mathfrak{L}^2(\mathbb{R}^n)$$

and the relations (9) take the form

$$(14) \quad \begin{aligned} US_{\pm}U^{-1} &= I \otimes P_{\pm}, & US_{j,\pm}U^{-1} &= (x_j/\sigma) \otimes Q_{\pm}, \\ U\Lambda_{\pm}U^{-1} &= (1/\sigma) \otimes Q_{\pm}. \end{aligned}$$

In (13) $\hat{\otimes}$ denotes the topological tensor product.

4. We notice that the operators P_{\pm}, Q_{\pm} of §1 are evidently in the algebra \mathfrak{F} as introduced in [4, §5]. In particular, Q_+ is a compact operator of \mathfrak{h} , Q_- is an even Wiener-Hopf convolution with \mathfrak{L}^1 -kernel, and P_{\pm} differ from cK_{\pm}^0 , with the operators K_{\pm}^0 as in [4] and a suitable constant c , only by a compact operator each. It is also easily seen that \mathfrak{F} may be generated as a C^* -algebra with unit by $\mathfrak{C}(\mathfrak{h}), P_{\pm}, Q_{\pm}$ as well as by the generators listed in [4].

DEFINITION. (a) \mathfrak{G}^{\sharp} denotes the C^* -subalgebra of $\mathfrak{R}(\mathfrak{F})$ without unit generated by the operators of the form

$$(15) \quad U^*(a(x) \otimes C)U$$

with $a(x) \in \mathfrak{C}(\mathbb{B}^n), C \in \mathfrak{C}(\mathfrak{h})$.

(b) $\mathfrak{A}^\#$ denotes the C^* -algebra with unit generated by $\mathfrak{G}^\#$ above and all operators $S_\pm, S_{j,\pm}, \Lambda_\pm, j=1, \dots, n$.

Note. As in [5] B^n denotes the smallest compactification of R^n into which the mapping $\rho: R^n \rightarrow \{|x| < 1\}$ defined by

$$(16) \quad \rho(x) = (2/\pi)x/|x| \arctan |x|, \quad x \neq 0, \quad = 0 \quad \text{for } x = 0$$

can be continuously extended.

We then have

THEOREM 1. $\mathfrak{G}^\#$ is a closed two-sided ideal of the C^* -algebra $\mathfrak{A}^\#$. The algebra $\mathfrak{A}^\#/\mathfrak{G}^\#$ is commutative and isometrically isomorphic to the function algebra $\mathcal{C}(\mathfrak{M}^\#)$ with the compact Hausdorff space $\mathfrak{M}^\#$ obtained from the product $B^n \times \mathfrak{M}(\mathfrak{F})$ by identifying all points of B^n over each point of the straight line segment $x=0, -\infty < t < +\infty, \xi = \infty$ in the space $\mathfrak{M}(\mathfrak{F})$ as defined in [4, §5].

Clearly $\mathfrak{G}^\#$ does not contain compact operators, except 0. On the other hand, $\mathfrak{G}^\#$ is contained in the R -algebra $U^*(\mathcal{C}(\mathfrak{F}) \hat{\otimes} \mathcal{L}(\mathfrak{h}))U = \mathfrak{F}$ and Theorem 1 relates the \mathfrak{F} -Fredholm property of $A \in \mathfrak{A}^\#$ to the non-vanishing of a continuous function over $\mathfrak{M}^\#$. (See [2], [3].)

Note that $M^\#$ is homeomorphic to an $n+1$ -ball B^{n+1} with the end-points of a one-dimensional interval I^1 attached to it at two distinguished points.

5. Let H^{n+1} denote the closure of R_+^{n+1} in B^{n+1} . It then is an easy consequence of results published in [5] that the commutators $[S_\pm, b], [S_{j,\pm}, b], [\Lambda_\pm, b], j=1, \dots, n$ are all in $\mathcal{C}(\mathfrak{F})$, for $b \in \mathcal{C}(H^{n+1})$.

DEFINITION. (a) \mathfrak{E} denotes the C^* -algebra without unit generated by $\mathcal{C}(\mathfrak{F})$ and all products $bE, Eb, b \in \mathcal{C}(H^{n+1}), E \in \mathfrak{G}^\#$.

(b) \mathfrak{A} denotes the C^* -algebra with unit generated by $\mathcal{C}(\mathfrak{F}), \mathfrak{A}^\#$ and $C(H^{n+1})$.

We then have the following main result.

THEOREM 2. (a) $\mathfrak{E} \subset \mathfrak{A}$ is a closed two-sided ideal of \mathfrak{A} , and $\mathfrak{A}/\mathfrak{E}$ is commutative.

(b) $\mathfrak{C} = \mathcal{C}(\mathfrak{F})$ is a closed two-sided ideal of E .

(c) The Gelfand space \mathfrak{M} of $\mathfrak{A}/\mathfrak{E}$ is (homeomorphic to) the following subset of the cartesian product $\mathfrak{M}^\# \times H^{n+1}$ ($\mathfrak{M}^\#$ as in Theorem 1):

(i) Over the boundary at $y = \infty$ of H^{n+1} one gets all points of $B^{n+1} \subset \mathfrak{M}^\#$.

(ii) Over interior points of $R_+^{n+1} \subset H^{n+1}$ one gets the boundary ∂B^{n+1} of the ball $B^{n+1} \subset \mathfrak{M}^\#$.

(iii) Over the boundary $y = 0$ of $R_+^{n+1} \subset H^{n+1}$ one gets the interval I^1 and the boundary ∂B^{n+1} of the ball $B^{n+1} \subset H^{n+1}$.

(iv) Over the points $y=0, |x| = \infty$ of H^{n+1} one gets the whole space $\mathfrak{M}^\#$.

(d) The algebra $\mathfrak{C}/\mathfrak{C}$ is isometrically isomorphic to the algebra $\mathcal{C}(\mathfrak{M}_1, \mathfrak{C}(\mathfrak{h}))$ of all continuous functions from a compact Hausdorff-space \mathfrak{M}_1 to the compact ideal $\mathfrak{C}(\mathfrak{h})$ of the Hilbert-space \mathfrak{h} .

(e) The space \mathfrak{M}_1 is (homeomorphic to) the set

$$(17) \quad \partial B^n \times B^n \cup B^n \times \partial B^n \subset B^n \times B^n,$$

(i.e., topologically is a $2n-1$ sphere).

DEFINITION. (a) To any $A \in \mathfrak{A}$ we associate $\sigma_A \in \mathcal{C}(\mathfrak{M})$ defined as the function associated to the coset of $A \bmod \mathfrak{C}$ by the Gelfand isomorphism of $\mathfrak{A}/\mathfrak{C}$. σ_A will be called the \mathfrak{C} -symbol of $A \in \mathfrak{A}$.

(b) To any $E \in \mathfrak{E}$ we associate $\tau_E \in \mathcal{C}(\mathfrak{M}_1, \mathfrak{C}(\mathfrak{h}))$ defined as image of the coset of $E \bmod \mathfrak{C}(\mathfrak{h})$ under the isomorphism (d) of Theorem 2. τ_E will be called the \mathfrak{C} -symbol of $E \in \mathfrak{E}$.

THEOREM 3. (a) A necessary condition for $A \in \mathfrak{A}$ to be Fredholm is that its \mathfrak{C} -symbol does never vanish on \mathfrak{M} .

(b) $A \in \mathfrak{A}$ with $\sigma_A \neq 0$ on \mathfrak{M} possesses an inverse $B \in \mathfrak{A} \bmod \mathfrak{C}$ such that $1-AB, 1-BA \in \mathfrak{C}$.

(c) $A \in \mathfrak{A}$ with $\sigma_A \neq 0$ is Fredholm if and only if for some \mathfrak{C} -inverse B of A we have

$$(1 + \tau_{(AB-1)}(m)) \text{ a regular operator of } \mathfrak{L}(\mathfrak{h})$$

for all $m \in \mathfrak{M}_1$.

6. The proof of Theorem 2 rests on the following facts partly of independent interest.

THEOREM 4. If $b \in C(H^{n+1})$ vanishes on the boundary $y=0$ then Eb, bE are compact, for all $E \in \mathfrak{E}^\#$.

The result of Theorem 4 may be expressed by saying that all operators of \mathfrak{E} are "compact, except over the boundary of R^{n+1} ."

THEOREM 5. We have $U\mathfrak{E}U^* = \mathfrak{C} \hat{\otimes} \mathfrak{C}(\mathfrak{h})$ with the algebra \mathfrak{C} as in [5, appendix].

This completely clarifies the structure of the ideal \mathfrak{C} and assertions (d) and (e) of Theorem 2 become evident, in view of [5] and [1].

While the proof of assertions (a) and (b) is a verification only, one may employ techniques as in [5] to obtain the precise extent of the space \mathfrak{M} .

We notice that the operators of our algebra \mathfrak{A} are similar to those considered by Vishik and Eskin [7], for instance.

Applicability of our results should strongly depend on the explicit construction of inverses mod \mathfrak{C} and of Fredholm inverses.

Especially we also expect results concerning pseudo-differential operators involving boundary conditions in a half-space like those in [4, §6].

REFERENCES

1. M. Breuer and H. O. Cordes, *On Banach algebras with σ -symbol*, Part II, J. Math. Mech. **14** (1965), 299–314.
2. H. O. Cordes, *On a class of C^* -algebras*, Math. Ann. **170** (1967), 283–313.
3. ———, *Über eine nichtalgebraische Charakterisierung von \mathfrak{S} -Fredholmoperatoren*, Math. Ann. **163** (1966), 212–229.
4. ———, *Pseudo-differential operators on half-line*, J. Math. Mech. (to appear).
5. H. O. Cordes and E. Herman, *Remarks on the Gel'fand theory of pseudo-differential operators*, Amer. J. Math. (to appear).
6. W. Magnus and F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen der Mathematischen Physik*, Springer, Berlin, 1948.
7. M. I. Vishik and G. I. Eskin, *Equations in convolutions in a bounded domain*, Uspehi Mat. Nauk **20** (1965), 90–152 = Russian Math. Surveys **20** (1965), 85–151.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720