ON GAUSSIAN SUMS

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This note is an outline of some of the author's recent work on a generalization of Fourier transforms in adele spaces. Here we treat only the simplest case. The details and a generalization for an arbitrary ground $\boldsymbol{A}$-field and a system of polynomials will be given elsewhere. For the unexplained notions, see [1], [2] and [3].

Let $f(X)$ be an absolutely irreducible polynomial in $\boldsymbol{Q}[X] = \boldsymbol{Q}[X_1, \cdots, X_n]$ such that the corresponding hypersurface $H = \{x \in \Omega^n; f(x) = 0\}$ is nonsingular, where Ω denotes a universal domain containing $\boldsymbol{Q}$. Let V be the complement of H in Ω^n viewed as an algebraic variety in Ω^{n+1} in an obvious way. Hence the n-form $\omega = f^{-1}dx$, $dx = dx_1 \wedge \cdots \wedge dx_n$, is everywhere holomorphic and never zero on V. For each valuation v of $\boldsymbol{Q}$, denote by $\boldsymbol{Q}_v$ the completion of $\boldsymbol{Q}$ at v. Denote by $\boldsymbol{A}$, $\boldsymbol{A}^*$ the adele ring and the idele group of $\boldsymbol{Q}$, respectively. For an idele $a \in \boldsymbol{A}^*$, $|a|_{\boldsymbol{A}}$ will denote the module of a. The adelization $V_{\boldsymbol{A}}$ of V is then given by $V_{\boldsymbol{A}} = \{x \in \boldsymbol{A}^n; f(x) \in \boldsymbol{A}^*\}$. We denote by $\mathcal{S}(\boldsymbol{Q}_v^n)$, $\mathcal{S}(\boldsymbol{A}^n)$ the space of Schwartz functions on $\boldsymbol{Q}_v^n$, $\boldsymbol{A}^n$, respectively. For each v, the n-form ω on V induces a measure ω_v on $V_{\boldsymbol{Q}_v}$ and we know that there is a well-defined measure $dV_{\boldsymbol{A}}$ on $V_{\boldsymbol{A}}$ of the form $\prod_v \lambda_v^{-1}\omega_v$ with $\lambda_\infty = 1$ and $\lambda_p = 1 - p^{-1}$. We know that the function

$$(1) \qquad Z(f, \phi, s) = \int_{V_{\boldsymbol{A}}} \phi(x) \, |f(x)|_{\boldsymbol{A}}^s dV_{\boldsymbol{A}}, \quad \phi \in \mathcal{S}(\boldsymbol{A}^n),$$

represents a meromorphic function for Re $s > \frac{1}{2}$ having the single simple pole at $s = 1$ with the residue $\int_{\boldsymbol{A}^n} \phi(x) d\boldsymbol{A}^n$, where $d\boldsymbol{A}^n$ is the canonical measure on $\boldsymbol{A}^n$ (cf. [4]).

Let χ be a basic character of $\boldsymbol{A}$ which identifies the additive group $\boldsymbol{A}$ with its own dual and let χ_v be the similar character of the additive group $\boldsymbol{Q}_v$ induced by χ. For each $\xi \in \boldsymbol{A}$ and $\phi \in \mathcal{S}(\boldsymbol{A}^n)$, the function $\phi_\xi(x) = \phi(x)\chi(f(x)\xi)$ is again in $\mathcal{S}(\boldsymbol{A}^n)$ and hence we have

$$(2) \qquad \operatorname*{Res}_{s=1} Z(f, \phi_\xi, s) = \int_{\boldsymbol{A}^n} \phi(x)\chi(f(x)\xi) d\boldsymbol{A}^n \overset{\text{def.}}{=\!=\!=} \mathcal{G}_f\phi(\xi).$$

The transform $\phi \to \mathcal{G}_f\phi$ is a linear map of $\mathcal{S}(\boldsymbol{A}^n)$ into the space of continuous functions on A, which boils down to the Fourier transform $\phi \to \mathfrak{F}\phi$ when $n=1$ and $f(X)=X$.[1]

Now, put $\eta = \operatorname{Res}\omega$, the residue form of ω, this being an $(n-1)$-form on the hypersurface H everywhere holomorphic and never zero. When the formal product $\prod_v \eta_v$ (with no convergence factors) really defines a measure on H_A, we say that the canonical measure $dH_A = \prod_v \eta_v$ exists on H_A. The classical theory of trigonometric sums suggests, at least formally, the equality

$$\int_A \mathcal{G}_f \phi dA = \int_{H_A} \phi dH_A \quad \text{for all } \phi \in \mathcal{S}(A^n), \tag{3}$$

where the right-hand side is essentially the singular series for $f(X)$ including the gamma factor. In view of (1), (2), one can interpret (3) as an equality connecting integrals on V_A and H_A. More precisely, one can prove the following

THEOREM 1. *If $f(X)$ satisfies the condition*

$$\mathcal{G}_f \phi \in L^1(A) \quad \textit{for all } \phi \in \mathcal{S}(A^n), \tag{C}$$

then the canonical measure dH_A exists, $\phi | H_A \in L^1(H_A)$ for all $\phi \in \mathcal{S}(A^n)$ and (3) *holds.*

Thus, the real problem is to find conditions on $f(X)$ so that (C) holds. For example, (C) is false for $f(X) = X_1^2 + \cdots + X_n^2$ with $n \leqq 4$. Although we are still far from the complete solution of the problem, we can give the following sufficient conditions.

THEOREM 2. *The condition* (C) *holds if $f(X)$ satisfies the following two conditions:*

(I) $\operatorname{grad} f(x) \neq 0$ *for all $x \in \Omega^n$,*

(II) $\left| \sum_{x \in F_p^n} \zeta_p^{f^{(p)}(x)} \right| \leqq c p^{n-2-\epsilon}$ *for almost all p, where c, ϵ are positive constants independent of p, $f^{(p)}(X)$ is the polynomial over the finite prime field F_p obtained by reducing the coefficients of $f(X)$, for almost all p, and ζ_p is any one of the primitive pth roots of* 1.

For example, $f(X) = X_1^{r_1} X_2^{r_2} + X_1 + \sum_{i=3}^n X_i^{r_i}$, $r_j \geqq 2$, $1 \leqq j \leqq n$, satisfies (I), (II) whenever $n \geqq 7$.

REMARK 1. The condition (I) implies that $\operatorname{grad} f(x) \neq 0$ for all $x \in Q_v^n$, for all v. For this case one proves the stronger result:

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[1] Such a transform has been introduced by Weil in more general setting [3, Chapter I, No. 1]. For p-adic case, the evaluation of the transform is substantially the Gaussian sum for the polynomial $f(X)$.

$$\mathcal{G}_f\phi(\xi) = \int_{Q_v^n} \phi(x)\chi_v(f(x)\xi)dx_v \in \mathcal{S}(Q_v) \quad \text{for all } \phi \in \mathcal{S}(Q_v^n).$$

This fact for $v=\infty$ has been suggested to us by Hörmander. We then found that the same is true for $v=p$.

REMARK 2. Unfortunately, the diagonal polynomial $f(X) = \sum_{i=1}^n a_i X_i^{r_i}$ does not, in general, satisfy (I). A direct verification of the condition (C) for such a polynomial seems to be not easy because arbitrary Schwartz functions are involved. However, (I) is intrinsic and this might be the case which must precede any attempt at a general theory.

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