

A FACTOR THEOREM FOR FRÉCHET MANIFOLDS

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1. Introduction. A *Fréchet manifold* (or *F-manifold*) is a separable metric space M having an open cover of sets each homeomorphic to an open subset of the countable infinite product of open intervals, s . A *Q-manifold* is a separable metric space M having an open cover of sets each homeomorphic to an open subset of the Hilbert cube, I^∞ . It is known that all separable metric Banach manifolds modeled on separable infinite-dimensional Banach spaces are *F-manifolds*. The following are the principle theorems of this paper.

THEOREM I. *If M is any F-manifold, then $s \times M$ is homeomorphic to M .*

THEOREM II. *If M is any Q-manifold, then $I^\infty \times M$ is homeomorphic to M .*

Since s is known, [1] or [3], to be homeomorphic to $s \times I^\infty$, from Theorem I we immediately have the following.

COROLLARY. *If M is any F-manifold, then $I^\infty \times M$ is homeomorphic to M .*

Almost identical proofs of Theorems I and II can be given. To emphasize the ideas of our proofs of Theorems I and II we shall outline instead a proof of the similar but notationally easier

THEOREM I'. *If M is any F-manifold and J^0 is the open interval $(-1, 1)$, then $J^0 \times M$ is homeomorphic to M .*

2. Lemma 2.1 implies Theorem I'.

DEFINITION. Let r be a map, i.e. continuous function, of a space X into the closed unit interval $[0, 1]$. Let $J^0(0) = \{0\}$ and for $t \in (0, 1]$, let $J^0(t) = (-t, t)$. Then $J^0 \times^r X = \{(y, x) \in J^0 \times X : y \in J^0(r(x))\}$ is the *variable product* of J^0 by X (with respect to r).

LEMMA 2.1. *Let U be an open subset of s , let $V \subset W \subset U$ where W is open and V is closed in U , and let $J^0 \times^{r_0} U$ be a variable product of J^0*

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by U . There exists a homeomorphism H of $J^0 \times^{r_0} U$ onto a variable product $J^0 \times^r U$ such that (1) $r \leq r_0$, (2) $r(V) = 0$, and (3) $H|_{J^0 \times^{r_0} [(U \setminus W) \cup r_0^{-1}(0)]}$ is the identity.

OUTLINE OF A PROOF THAT LEMMA 2.1 IMPLIES THEOREM I'. Since M is separable and metric, there exists a countable star-finite open cover G of M with sets homeomorphic to open subsets of s . (By star-finite cover we mean a cover such that the closure of each element intersects only finitely many closures of other members of the cover.) Thus if $V \subset U$ where $U \in G$ and V is closed in M and $J^0 \times^{r_0} M$ is a variable product of J^0 by M , then by taking an open set W in M such that $W \supset V$ and $\text{Cl}_M W \subset U$, Lemma 2.1 will imply the existence of a homeomorphism of $J^0 \times^{r_0} U$ that has an automatic extension to a homeomorphism H of $J^0 \times^{r_0} M$ where (1) $r \leq r_0$, (2) $r(V) = 0$, and (3) $H|_{J^0 \times^{r_0} [(M \setminus U) \cup r_0^{-1}(0)]}$ is the identity.

As suggested by Theorem 2 of [2], we take a special ordering of the elements of G , say $\{U_i\}_{i>0}$. Now take a cover $\{V_i\}_{i>0}$ of M where, for each $i > 0$, V_i is a closed set contained in U_i . For each $i > 0$, let H_i be a homeomorphism from $J^0 \times^{r_{i-1}} M$ onto $J^0 \times^{r_i} M$ where $r_i \leq r_{i-1} \leq \dots \leq r_0 = 1$, $r_i(V_i) = 0$, and $H_i|_{J^0 \times^{r_{i-1}} (M \setminus U_i)}$ is the identity. Then $(H_i \circ \dots \circ H_1)_{i>0}$ converges to a homeomorphism of $J^0 \times M$ onto $\{0\} \times M$ which is homeomorphic to M .

3. Two lemmas leading to Lemma 2.1. Let $s = \prod_{i>0} J_i^0$ where for each $i > 0$, $J_i^0 = J^0$. Let $\pi: J^0 \times s \rightarrow s$ be the natural projection onto s and for $n > 0$, let π_n be defined on s as follows. For $z = (z_1, z_2, \dots) \in s$, let $\pi_n(z) = (z_1, \dots, z_n, 0, 0, \dots)$. Also, for Y a space and $f: s \rightarrow Y$, define $f^*: J^0 \times s \rightarrow Y$ by $f^* = f\pi$.

LEMMA 3.1. There exists a map

$$h: (J^0 \times s) \times [0, 1] \times [1, \infty) \rightarrow J^0 \times s$$

such that if $t \in [0, 1]$ and $u \in [1, \infty)$ are fixed where $n \leq u$, the map

$$H: J^0 \times s \rightarrow J^0 \times s$$

defined by $H(p) = h(p, t, u)$ for $p \in J^0 \times s$ is a homeomorphism of $J^0 \times s$ onto $J^0 \times^r s$ where (1) $r = 1 - t$, (2) if $t = 0$, H is the identity, and (3) $\pi_n^* = \pi_n^* H$.

OUTLINE OF PROOF. It suffices to describe h . We first describe for any integer $n > 0$ a map h' of $(J^0 \times s) \times [0, 1] \times \{n\}$ onto $J^0 \times s$.

For $x = (x_0, x_1, \dots) \in J^0 \times s$, let $h'(x, 0, n) = x$ and as t varies from 0 to $\frac{1}{2}$ let the 0th and $(n+1)$ th coordinates of x be "rotated" so that at $t = \frac{1}{2}$, (x_0, x_{n+1}) becomes $(x_{n+1}, -x_0)$ while for $0 \leq t \leq \frac{1}{2}$ all other coordinates are left fixed.

Thus, $h'(x, \frac{1}{2}, n) = (x_{n+1}, x_1, \dots, x_n, -x_0, x_{n+2}, \dots)$. For $i > 0$, as t varies from $1 - 2^{-i}$ to $1 - 2^{-(i-1)}$, "rotate" the 0th and $(n+i+1)$ th coordinates leaving all other coordinates fixed so that for each $i > 0$,

$$h'(x, 1 - 2^{-i}, n) = (x_{n+i}, x_1, \dots, x_n, -x_0, -x_{n+1}, \dots, -x_{n+i-1}, x_{n+i+1}, \dots).$$

To define h from h' we specify that $h(x, 1, n) = (0, x_1, \dots, x_n, -x_0, -x_{n+1}, -x_{n+2}, \dots)$ and we introduce for any time $t < 1$, a factor of $1-t$ in the 0th coordinate place of h' . Indeed for each integer $n \geq 1$, $h|(J^0 \times s) \times [0, 1] \times \{n\}$ becomes an isotopy. It is now possible to extend the domain of h for values of u between n and $n+1$ by use of "rotations" similar in nature to the "rotations" used in defining h' for a fixed n .

An open set E of s is an n -basic open set in s if $E = E_1 \times \dots \times E_n \times \prod_{i>n} J_i^0$ where each E_i is open in J_i^0 and is a subinterval of J_i^0 .

DEFINITION. Let W be open in s and let $\{G_i\}$ be a star finite collection of m_i -basic open sets in s whose union is W . For each $x \in W$ let

$$m_x = \text{minimum}\{m_i: x \in G_i\}.$$

Let Y be a space. A map $f: W \rightarrow Y$ is a *local product map* of W with respect to the G_i and m_i if $f(x) = f(\pi_{m_x}(x))$ for each $x \in W$. If, additionally, $Y = [1, \infty)$ and $f(x) \geq m_x$ for each $x \in W$, then f is a *local product indicator map* of W with respect to the G_i and the m_i .

The strategy is to replace the t and u of Lemma 3.1 by local product maps. The following technical lemma (not proved here) asserts the existence of the proper type of local product maps.

LEMMA 3.2. *Let U be an open subset of s and let $V \subset W \subset U$ and $A \subset U$ where W is open and V and A are closed in U . There exist a countable star finite collection $\{G_i\}$ of m_i -basic open sets in s whose union is $W \setminus A$ and maps $\phi: U \setminus A \rightarrow [0, 1]$ and $g: W \setminus A \rightarrow [1, \infty)$ such that (1) $\phi(V \setminus A) = 1$, (2) $\phi((U \setminus W) \setminus A) = 0$, (3) $\phi|W \setminus A$ and g are a local product map and a local product indicator map, respectively, of $W \setminus A$ with respect to the G_i and m_i , and (4) g is unbounded near A , that is, for any $x \in A \cap \text{Cl}(W \setminus A)$ and $n > 0$, there is a neighborhood $B(x)$ such that $g|(W \setminus A) \cap B(x) > n$.*

4. **Proof of Lemma 2.1.** By Lemma 3.2 take a star finite collection $\{G_i\}$ of m_i -basic open sets and the maps ϕ and g for the case when $A = r_0^{-1}(0)$. Now, let

$$h: (J^0 \times s) \times [0, 1] \times [1, \infty) \rightarrow J^0 \times s$$

be the map of Lemma 3.1 and define

$$H_1: J^0 \times W \setminus r_0^{-1}(0) \rightarrow J^0 \times s$$

by $H_1(p) = h(p, \phi^*(p), g^*(p))$ for $p \in J^0 \times W \setminus r_0^{-1}(0)$. It can be shown that H_1 is a homeomorphism onto $J^0 \times r_1 W \setminus r_0^{-1}(0)$ where $r_1 = 1 - \phi$. Clearly the map k from $J^0 \times r_0 W \setminus r_0^{-1}(0)$ to $J^0 \times W \setminus r_0^{-1}(0)$ defined by $k(y, z) = (y r_0^{-1}(z), z)$ is a homeomorphism. Also $k^{-1} H_1 k$ is a homeomorphism from $J^0 \times r_0 W \setminus r_0^{-1}(0)$ onto $J^0 \times r W \setminus r_0^{-1}(0)$ where $r = (1 - \phi) r_0$. Now define $H: J^0 \times r_0 U \rightarrow J^0 \times r U$ by $H = k^{-1} H_1 k$ on $J^0 \times r_0 W \setminus r_0^{-1}(0)$ and $H = \text{identity}$ on $J^0 \times [(U \setminus W) \cup r_0^{-1}(0)]$. We show that H is continuous. Since $\phi((U \setminus W) \setminus r_0^{-1}(0)) = 0$, from condition 2 of Lemma 3.1 it follows that $H|_{J^0 \times (W \setminus r_0^{-1}(0))}$ and the identity map on $J^0 \times [(U \setminus W) \setminus r_0^{-1}(0)]$ are compatible. To show that these are compatible with the identity $J^0 \times r_0 r_0^{-1}(0)$ we check the coordinatewise continuity of H . The continuity of r_0 gives the continuity of H on the first, or J^0 , coordinate and g becoming unbounded near $r_0^{-1}(0)$ yields the continuity of H on the second, or U , coordinate. The other conditions of Lemma 2.1 are easily seen to be satisfied.

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