PERIODIC ORBITS OF HYPERBOLIC DIFFEOMORPHISMS AND FLOWS[1]

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Artin and Mazur in [1] proved that a dense subset of the C^r-endomorphisms of a compact differentiable manifold satisfy an exponential growth condition on their isolated periodic points, and they defined a ζ-function which for these endomorphisms has a positive radius of convergence. In [2] and [3] K. Meyer gave a simple proof that hyperbolic diffeomorphisms and flows of Smale [4] which are C^2 have exponential growth. It is the purpose of this note to give an even simpler proof of Meyer's theorems in a C^1 setting. Since the hyperbolic diffeomorphisms and flows are not dense [5] these results are a long way from including the results of [1].

Let M be a compact differentiable manifold; let $f \in \mathrm{Diff}(M)$ be a C^1 diffeomorphism, and let $N_m(f)$ be the number of periodic points of f of period m.

THEOREM 1. *Let f satisfy Axiom* A *of* [4, I.6], *then there exist constants c and k such that $N_m(f) \leqq ck^m$.*

PROOF. f is expansive [4, I.8.7], i.e., $\exists \epsilon > 0$ such that given x, y distinct periodic points of $f \exists n \in Z$ such that $d(f^n(x), f^n(y)) \geqq \epsilon$. Since f is C^1 it is Lipshitz. Let its Lipshitz constant be k which we may choose > 1. If x and y are both of period p we may choose n in $0 \leqq n < p$ and have $d(x, y) \geqq \epsilon / k^{p-1}$ by expansiveness. Thus there exists a constant c such that $N_p(f) \leqq c V(M)(2k^{p-1}/\epsilon)^{\dim M}$ where $V(M)$ is the volume of M.

Let $\Phi = \{\phi_t\}$ be a one parameter group acting on M, arising from a C^1 vector field X. Let $N_\tau(\Phi)$ be the number of closed orbits of Φ of period less than or equal to τ.

THEOREM 2. *Let Φ satisfy Axiom* A′ *of* [4, 5.1], *then there exist constants c and k such that $N_\tau(\Phi) \leqq ce^{k\tau}$.*

Since the closed orbits are uniformly bounded away from the singularities, which are finite in number, Ω_c the complement of the singularities in Ω, is compact. Every point z in Ω_c has a flow box

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neighborhood, V_z, such that V_z has a cross section X_z; and for some δ, $V_z = \bigcup_{x \in X_z, |t| < \delta} \phi_t(x)$ where δ is independent of z. Denote by π_z: $V_z \to X_z$ the map which takes $\phi_t(x)$ to x for $|t| < \delta$ and $x \in X_z$. The V_z and t may be chosen so that if $x, y \in X_z$ and $\phi_t(x), \phi_t(y) \in X_{w_1}$; $\phi_{-t}(x), \phi_{-t}(y) \in X_{w_2}$ then either $\pi_{w_1}\phi_t$ or $\pi_{w_2}\phi_{-t}$ increases their distance by a factor of $k_1 > 1$; k_1 independent of z.

Now cover Ω_c by a finite number of V_z; $V_1, V_2, \cdots, V_n$. Note that if $x \in X_i$ and $\phi_t(x) \in V_j$ then $\pi_i\phi_{-t}\pi_j\phi_t(x) = x$, and similarly for ϕ_{-t}. Define an invariant sequence of a closed orbit $\alpha \subset \Omega_c$ as a sequence $a_1 \cdots a_m$ such that $a_i = 1, \cdots, n$; $a_1 = a_m$ and there exist $x_{a_i} \in X_{a_i}$ where $x_{a_1} = x_{a_m}$, $\pi_{a_i}\phi_t(x_{a_{i-1}}) = x_{a_i}$ and $\pi_{a_{i-1}}\phi_{-t}(x_{a_i}) = x_{a_{i-1}}$. It is clear from the definition and the choice of the V_i that no two distinct closed orbits may have the same invariant sequence. We will show that $\exists c > 0$ such that a closed orbit of period $\leqq \tau$ has an invariant sequence of length at most $nc\tau + 1$, and thus the number of closed orbits of period $\leqq \tau$ is less than or equal to $n^{nc\tau+2} = n^2 e^{\tau c \log n}$.

To get the invariant sequence, $\exists c > 0$ such that if α is a closed orbit of period $\leqq \tau$ then α intersects X_i for any i in at most $c\tau$ points, c is independent of τ. So there are at most $nc\tau$ intersections of α with all the X_i. Let x_0 be one of these. Define x_i inductively by $x_{i+1} = \pi_j\phi_t(x_i)$ where $\phi_t(x_i) \in V_j$ for some j. x_k must equal x_m for some $k, m \leqq nc\tau$ and $k \neq m$.

Of course, this proof would also work in the diffeomorphism case. The construction of the corresponding V_i, however, essentially gives the proof of expansiveness which was shown to me by Smale.

These theorems, of course, have relevance for the convergence of the zeta functions. For a discussion of this see [3] and [4].

References

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