MANIFOLDS HOMEOMORPHIC TO SPHERE BUNDLES OVER SPHERES

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1. Statement of results. Let E be the total space of a k-sphere bundle over the n-sphere with characteristic class $\alpha \in \pi_{n-1}(SO_{k+1})$. We consider the problem of classifying, under the relation of orientation preserving diffeomorphism, all differential structures on E. It is assumed that E is simply connected, of dimension greater than five, and its characteristic class α may be pulled back to lie in $\pi_{n-1}(SO_k)$ (that is, the bundle has a cross-section). In [1] and [2] we gave a complete classification in the special case where $\alpha = 0$. The more general classification Theorems 1 and 2 below include this special case. The proofs of these theorems are sketched in §2 below; detailed proofs will appear elsewhere. J. Munkres [6] has announced a classification up to concordance of differential structures in the case where the bundle has at least two cross-sections. (It is well known that concordance and diffeomorphism are not equivalent, concordance of differential structures being strictly stronger than diffeomorphism.)

THEOREM 1. *Let E be the total space of a k-sphere bundle over the n-sphere whose characteristic class*[2] *α may be pulled back to lie in $\pi_{n-1}(SO_k)$. Suppose that $2 \leqq k < n-1$. Then, under the relation of orientation preserving diffeomorphism, the diffeomorphism classes of manifolds homeomorphic to E are in a one-to-one correspondence with the equivalence classes on the set $(\theta_n/\Phi_n^{k+1}) \times \theta_{n+k}$, where (A_*^n, U^{n+k}) and (B_*^n, V^{n+k}) are equivalent if and only if $A_*^n = \pm B_*^n$ and there exists $\beta \in \pi_k(SO_{n-1})$ such that $U^{n+k} - V^{n+k} = \tau'_{n,k}(A_*^n \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta)$.*

Theorem 1 is also true in the case where $k = n-1$ and n is odd. The classification in the case where $n-1 \leqq k \leqq n+2$ is essentially the same as the above and is given in Theorem 2 below. Now we establish the notation used in Theorem 1.

NOTATION. Manifolds and diffeomorphisms are of class C^∞. The group of homotopy n-spheres under the connected sum operation +

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[2] *Added in proof.* Assume here and in Proposition 2 that α is of order 2 in $\pi_{n-1}(SO_{k+1})$ in the case where $k < n-3$. This assumption is not made elsewhere.

is denoted by θ_n, and Φ_n^{k+1} is the subgroup of θ_n consisting of those homotopy n-spheres that embed in $(n+k+1)$-space with a trivial normal bundle. The class of a homotopy n-sphere A^n in the group θ_n/Φ_n^{k+1} is denoted by A_*^n. Now let

$$\sigma_{n-1,k}: \pi_{n-1}(SO_k) \otimes \pi_k(SO_{n-1}) \to \theta_{n+k},$$
$$\tau_{n,k}: \theta_n \otimes \pi_k(SO_{n-1}) \to \theta_{n+k}$$

be the pairings defined in [4, p. 583]. It is known that these pairings correspond to composition in the stable homotopy groups of spheres. Moreover, it was shown in [2] that $\tau_{n,k}(\Phi_n^{k+1} \otimes \pi_k(SO_{n-1}))=0$, provided that $k \geqq 2$, and hence the pairing $\tau_{n,k}$ induces a pairing

$$\tau'_{n,k}: (\theta_n/\Phi_n^{k+1}) \otimes \pi_k(SO_{n-1}) \to \theta_{n+k} \qquad (k \geqq 2).$$

REMARK. If $k \geqq n-3$, then $\Phi_n^{k+1}=\theta_n$ (see [1, Lemma 1]) and hence $\tau_{n,k}=\tau'_{n,k}=0$ for $k \geqq n-3$.

In order to state the result in the case where $n-1 \leqq k \leqq n+2$ we define a function

$$\sigma'_{n,k}: \pi_{n-1}(SO_k) \times \pi_k(SO_n) \to \theta_{n+k} \tag{1}$$

that is linear in the second variable. The definition of $\sigma'_{n,k}$ is similar to the definition of the pairing $\sigma_{n-1,k}$ and is described in §2 below. Now if $\alpha \in \pi_{n-1}(SO_k)$ is the characteristic class of the bundle E, then we define a homomorphism

$$\chi_\alpha: \pi_k(SO_n) \to \theta_{n+k}$$

by writing, for each $\beta \in \pi_k(SO_n)$,

$$\chi_\alpha(\beta) = \sigma'_{n,k}(\alpha,\beta).$$

THEOREM 2. *Suppose that the characteristic class α of the bundle E may be pulled back to lie in $\pi_{n-1}(SO_k)$. Then, if $1 \leqq n-3 \leqq k \leqq n+2$ and $k \geqq 2$, then the diffeomorphism classes of manifolds homeomorphic to E are in a one-to-one correspondence with the group $\theta_{n+k}/\mathrm{Image}\ \chi_\alpha$.*

2. **Outline of proofs.** We give E the "standard" differential structure by making it a smooth k-sphere bundle over the standard n-sphere S^n. It is well known that if a k-sphere bundle over the n-sphere has a cross-section, then the total space of the bundle has the homology of the product $S^n \times S^k$. The proof of Theorem 1 is divided into the following four propositions. We use the notation $E(A^n)$ to denote the differential $(n+k)$-manifold obtained by making E into a smooth k-sphere bundle over a homotopy n-sphere A^n in the obvious way (if

$n=4$, then take A^4 to be homeomorphic to S^4). We assume that E has a cross-section, $n>3$, and $n+k>5$. We also assume that $k\geqq 2$, except in Proposition 4 where we allow $k=1$.

PROPOSITION 1. *If M is a differential $(n+k)$-manifold that is homeomorphic to E, then there are homotopy spheres A^n and U^{n+k} such that M is diffeomorphic to $E(A^n)+U^{n+k}$, provided that $k\leqq n+2$.*

SKETCH OF PROOF. Since E is of dimension greater than five and simply connected we can apply the Hauptvermutung of [7] to conclude that there is a PL-homeomorphism $h\colon M\rightarrow E$, where the combinatorial structures are compatible with the differential structures. We try to smooth h by applying the obstruction theory of Munkres [5]. If $k<n$, then the first obstruction to deforming h into a diffeomorphism is an element $c(h)$ in $H_n(M;\Gamma_k)$, where Γ_k is the group of diffeomorphisms of S^{k-1} modulo those that extend to diffeomorphisms of the k-disk D^k. Since $H_n(M;\Gamma_k)$ is isomorphic to Γ_k we can consider $c(h)$ to be an element of Γ_k. Now we construct a manifold $M(c(h))$ and a PL-homeomorphism j from E to $M(c(h))$ such that the first obstruction to smoothing j is $-c(h)$. It follows that the first obstruction to smoothing the composition jh is zero and hence we can suppose that jh is a diffeomorphism modulo the k-skeleton. The next step is to show that there is a diffeomorphism modulo a point $\phi\colon M(c(h))\rightarrow E$, (this is true for $k\leqq n+2$) and hence the composition $h'=\phi jh$ is a diffeomorphism modulo the k-skeleton. The first obstruction to smoothing $h'\colon M\rightarrow E$ is an element $c(h')$ in $H_k(M;\Gamma_n)\approx\Gamma_n$. Now let A^n be the homotopy n-sphere that corresponds to $c(h')$ under the isomorphism $\Gamma_n\approx\theta_n$ $(n\neq 3)$. There is a PL-homeomorphism j' from E to $E(A^n)$. Moreover, the first obstruction to smoothing j' is $-c(h')$ and hence we can assume that the composition $j'h'$ is a diffeomorphism up to a point. It follows that there is a homotopy $(n+k)$-sphere U^{n+k} such that M is diffeomorphic to $E(A^n)+U^{n+k}$. The argument in the case where $n\leqq k\leqq n+2$ is essentially the same. Note that if $n=4$, then the homotopy sphere A^4 is homeomorphic and hence diffeomorphic to S^4 since $\Gamma_4=0$.

The remaining propositions combine to give a classification of manifolds of the form $E(A^n)+U^{n+k}$.

PROPOSITION 2. *$E(A^n)$ and $E(B^n)$ are diffeomorphic if and only if $A^n\equiv\pm B^n \bmod \Phi_n^{k+1}$.*

The proof of Proposition 2 is similar to the proofs of Lemmas 5 and 6 of [1]. R. Schultz informs me that he has also proved Proposition 1 and Proposition 2.

PROPOSITION 3. *If $E(A^n)+U^{n+k}$ is diffeomorphic to $E(B^n)$, where A^n, B^n, U^{n+k} are homotopy spheres, then $E(A^n)$ and $E(B^n)$ are diffeomorphic.*

The proof of Proposition 3 is similar to the proof of Lemma 3 of [1].

PROPOSITION 4. *Let A^n, U^{n+k} be homotopy spheres such that $1\leqq k<n-1$. Then, $E(A^n)+U^{n+k}$ is diffeomorphic to $E(A^n)$ if and only if there exists an element $\beta\in\pi_k(SO_{n-1})$ such that*

$$U^{n+k} = \tau_{n,k}(A^n \otimes \beta) + \sigma_{n-1,k}(\alpha \otimes \beta).$$

The proof of Proposition 4 is similar to the proof of Theorem 3.1 of [2] except that the proof here is a bit more complicated since there are two pairings involved rather than just the pairing $\tau_{n,k}$.

Now we give the construction of the function $\sigma'_{n,k}$ of (1) in §1. Let $\gamma: S^{n-1}\rightarrow SO_k$ and $\beta: S^k\rightarrow SO_n$ be differentiable maps that represent elements in $\pi_{n-1}(SO_k)$ and $\pi_k(SO_n)$, respectively. We can assume that β maps the southern hemisphere D^k_- of S^k into the identity of SO_n. Define diffeomorphisms λ_γ and μ_β of $S^{n-1}\times S^k$ by writing, for each $(u, v)\in S^{n-1}\times S^k$, $\lambda_\gamma(u, v)=(u, s\gamma(u)\cdot v)$ and $\mu_\beta(u, v)=(\beta(v)\cdot u, v)$; here the dot denotes the action of the rotation group on the sphere and s denotes the natural inclusion of SO_k in SO_{k+1}. It is clear that $\lambda_\gamma(S^{n-1}\times D^k_-)=S^{n-1}\times D^k_-$ and hence it follows that the diffeomorphism $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ of $S^{n-1}\times S^k$ is the identity on $S^{n-1}\times D^k_-$. Now if B^{n+k} is an $(n+k)$-disk in the interior of $D^n\times S^k$, then it follows that the diffeomorphism $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ can be extended to a diffeomorphism of $D^n \times S^k$ − Interior B^{n+k}. The diffeomorphism induced on the $(n+k-1)$-sphere ∂B^{n+k} determines an element $\sigma'_{n,k}(\gamma, \beta)$ of θ_{n+k}, and it is not hard to show that this element depends only on the homotopy classes of γ and β. In fact, $\sigma'_{n,k}(\gamma, \beta)$ is the obstruction to extending $\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma$ to a diffeomorphism of $D^n\times S^k$. Since obstructions are additive with respect to compositions and

$$\lambda_\gamma^{-1}\mu_{\beta+\beta'}\lambda_\gamma = (\lambda_\gamma^{-1}\mu_\beta\lambda_\gamma)(\lambda_\gamma^{-1}\mu_{\beta'}\lambda_\gamma),$$

the correspondence $(\gamma, \beta)\rightarrow\sigma'_{n,k}(\gamma, \beta)$ is linear in β.

PROPOSITION 5. *Let A^n and U^{n+k} be homotopy spheres such that $1\leqq n-3\leqq k<2n-3$. Then, $E(A^n)+U^{n+k}$ is diffeomorphic to $E(A^n)$ if and only if there exists an element $\beta\in\pi_k(SO_n)$ such that $U^{n+k}=\chi_\alpha(\beta)$.*

Now Theorem 2 follows by applying Propositions 1, 2, and 5, noting that $\Phi^{k+1}_n=\theta_n$ for $k\geqq n-3$.

We conclude with some remarks on the case where $k>n+2$. Proposition 1 is not true in this case. For example let Σ^{16} denote the nonzero element of $\theta_{16} \approx Z_2$. It is known that Σ^{16} does not embed in R^{29} with a trivial normal bundle [3, Theorem 1.3]. Suppose that the conclusion of Proposition 1 is true for $S^{12} \times \Sigma^{16}$; that is, suppose that $S^{12} \times \Sigma^{16}$ is diffeomorphic to $(A^{12} \times S^{16}) + U^{28}$ for homotopy spheres A^{12} and U^{28}. It is well known that $A^{12} \times S^{16}$ is diffeomorphic to $S^{12} \times S^{16}$ and hence it follows that $S^{12} \times \Sigma^{16}$ and $S^{12} \times S^{16}$ are diffeomorphic up to a point. This implies that Σ^{16} embeds in R^{29} with a trivial normal bundle, a contradiction. On the other hand if $k > n+2$, then the characteristic class α may be pulled back to lie in $\pi_{n-1}(SO_{k-2})$. Define homomorphisms $\eta_\alpha: \theta_k \to \theta_{n+k-1}$ and $\eta'_\alpha: \theta_{k+1} \to \theta_{n+k}$ by writing

$$\eta_\alpha(\Sigma^k) = \tau_{k,n-1}(\Sigma^k \otimes \alpha) \quad \text{and} \quad \eta'_\alpha(\Sigma^{k+1}) = \tau_{k+1,n-1}(\Sigma^{k+1} \otimes \alpha)$$

for $\Sigma^k \in \theta_k$ and $\Sigma^{k+1} \in \theta_{k+1}$, respectively. It follows from [2] that $\Phi^n_k \subset \text{Kernel } \eta_\alpha$. Moreover, we can show that the number of distinct (nondiffeomorphic) differential structures on E is not greater than the order of Kernel η_α / Φ^n_k times the order of $\theta_{n+k}/\text{Image } \eta'_\alpha$. We plan to give the explicit computation at a later date. Finally, it follows from Munkres [6] that the concordance classes of differential structures on E are in a one-to-one correspondence with

$$\theta_n \oplus (\text{Kernel } \eta_\alpha) \oplus (\theta_{n+k}/\text{Image } \eta'_\alpha).$$

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