STRONGLY NEGLIGIBLE SETS IN
FRÉCHET MANIFOLDS

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Let \( s \) denote the linear metric space which is the countable infinite product of lines. It is known \([1]\) that \( s \) is homeomorphic to Hilbert space \( l_2 \) and, in light of \([8]\) and \([10]\), to all separable infinite-dimensional Fréchet spaces (and therefore, of course, to all such Banach spaces). We define a Fréchet manifold or \( F \)-manifold to be a separable metric space which admits an open cover by sets homeomorphic to open subsets of \( s \). Banach manifolds, which may be similarly defined, have been studied by a number of authors. From the results cited above it follows that all separable metric Banach manifolds modeled on separable infinite-dimensional Banach spaces are, in fact, \( F \)-manifolds. Also, clearly, any open subset of an \( F \)-manifold is an \( F \)-manifold.

In this paper, we are concerned with homeomorphisms of \( F \)-manifolds onto dense subsets of themselves. The first result of the type we consider was due to Klee \([11]\), who showed that for any compact set \( K \) in \( l_2 \), \( l_2 \) is homeomorphic to \( l_2 \setminus K \). Recently, there have been a number of results \([2]\), \([3]\), \([4]\), \([5]\), \([7]\), \([13]\), etc., showing that for various types of subsets \( K \) of certain linear metric spaces \( X \), \( X \) is homeomorphic to \( X \setminus K \). Bessaga \([7]\) introduced the term "negligible" for such sets \( K \). In some cases \( K \) was assumed compact, in others \( \sigma \)-compact (i.e. the countable union of compact sets) and in others \( K \) was assumed to be the countable union of closed sets of infinite deficiency (i.e. of infinite codimension). Indeed several different geometric methods \([2]\), \([3]\), \([5]\), \([7]\), \([11]\) have been used to establish negligibility in various spaces. The results that \( \sigma \)-compact subsets of \( l_2 \) and of \( s \) are negligible were used in the proofs \([1]\) and \([5]\) that \( l_2 \) is homeomorphic to \( s \). Questions of negligibility of subsets in Fréchet and Banach manifolds have also arisen. Where differentiable structures are assumed as for Banach spaces and manifolds and \( K \) is assumed closed, Bessaga \([7]\), Corson, Eells and Kuiper \([9]\), Kuiper and Burghelea \([12]\), Moulis \([13]\), Renz \([15]\) and West have investigated conditions under which \( X \) and \( X \setminus K \) are diffeomorphic.

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or they have used results of this type in other work. However, the results being announced in this paper are concerned only with homeomorphisms, not with diffeomorphisms.

In [6], Henderson, West, and the author introduced the concept of strong negligibility and characterized the strongly negligible closed subsets of an F-manifold. A subset $K$ of a space $X$ is \textit{strongly negligible} if for any open cover $G$ of $X$ there exists a homeomorphism $h$ of $X$ onto $X \setminus K$ such that $h$ is limited by $G$, i.e., for any $p \in X$ there exists $g \in G$ such that both $p$ and $h(p)$ are elements of $g$.

A similar concept related to the metric of a space is the concept of metric negligibility. A set $K$ in a metric space $X$ is \textit{metrically negligible} in $X$ if for each $\varepsilon > 0$, there exists a homeomorphism $h$ of $X$ onto $X \setminus K$ such that $h$ moves no point more than $\varepsilon$. Clearly, in a metric space $X$, strong negligibility of a set $K$ implies metric negligibility since we may select an open cover of $X$ of mesh less than $\varepsilon$. It is nontrivial, but follows from Theorem I below that, in an F-manifold, metric negligibility of a set $K$ implies strong negligibility of $K$.

Following [4], a closed set $K$ has \textit{Property Z} in a space $X$ if for each nonnull homotopically trivial open set $U$ in $X$, $U \setminus K$ is nonnull and homotopically trivial. (A set $U$ is homotopically trivial if every map of an $n$-sphere $S^n$, $n \geq 0$, into $U$ can be extended to a map into $U$ of an $(n+1)$-ball bounded by $S^n$.) In a sense, Property Z is "trivial homotopy negligibility." See [9] for a similar point-of-view.

The following theorem is proved in [6].

\textbf{Theorem 0.} A closed set $K$ in an F-manifold $X$ is strongly negligible \iff{} $K$ has Property Z.

It should be noted that every compact set in an F-manifold $X$ has Property Z in $X$, that every closed set of infinite deficiency in $s$ or in a separable metric Banach space has Property Z in such space, and that every closed set which is a countable union of closed sets with Property Z in an F-manifold $X$ has Property Z in $X$.

The principal result of this paper is the following theorem.

\textbf{Theorem I.} A set $K$ in an F-manifold $X$ is strongly negligible (or metrically negligible) in $X$ \iff{} $K$ is a countable union of closed sets with Property Z in $X$.

Theorem I includes, as special cases or easy corollaries, Theorem 0 and many or all of the previous results on negligibility in F-manifolds $X$ under homeomorphisms of $X$ onto dense subsets of itself.

The proof of necessity in Theorem I is fairly straightforward. We do not outline it here.
The proof of sufficiency depends heavily on the canonical compactification of \( s \) as the Hilbert cube \( I^\infty \) in which \( s \) is regarded as a product of open intervals and the Hilbert cube is regarded as the product of the closures of the open intervals. Thus \( I^\infty = \prod_{j>0} I_j \) and \( s = \prod_{j>0} I_j^\circ \) where for each \( j>0, I_j = [-1, 1] \) and \( I_j^\circ = (-1, 1) \). We let \( B(I^\infty) \) denote \( I^\infty \setminus s \). A set \( K \subseteq I^\infty \) is an apparent boundary of \( I^\infty \) if there exists a homeomorphism \( h \) of \( I^\infty \) onto \( I^\infty \) such that \( h(K) = B(I^\infty) \).

In [6], a rather general procedure for reducing certain homeomorphism problems on \( F \)-manifolds to homeomorphism problems on the Hilbert cube or on \( s \) itself is given. The actual homeomorphism theorems on \( I^\infty \) and \( s \) that are needed in [6] can be found in [2], [4], [5]. While we use the general procedures of [6] (with slight modifications) to establish sufficiency in Theorem I, we also use the following new homeomorphism theorem about \( I^\infty \).

**Theorem II.** Let \( I^\infty \supseteq K \supseteq B(I^\infty) \). Then \( K \) is an apparent boundary of \( I^\infty \) iff \( K \) is a countable union of closed sets with Property \( Z \) in \( I^\infty \).

In effect, Theorem II characterizes those apparent boundaries of \( I^\infty \) which contain \( B(I^\infty) \).

The sufficiency statement of Theorem II can be used to prove the somewhat stronger Theorem II A below, which is in a form more readily adaptable for application to \( F \)-manifolds. An endslice of \( I^\infty \) is a set \( W \) such that for some \( i>0, W = \{ (x_j) \in I^\infty | x_i = 1 \text{ (or } -1) \} \).

**Theorem II A.** Let \( W^* \) be a finite union of endslices in \( I^\infty \), let \( \varepsilon > 0 \), and let \( K \) be a countable union of closed sets with Property \( Z \) in \( I^\infty \) such that \( K \cap W^* = \emptyset \). Then there exists a homeomorphism \( h \) of \( I^\infty \) onto \( I^\infty \) such that \( h \mid W^* = \text{identity}, h(s \setminus K) = s, \) and \( h \) moves no point more than \( \varepsilon \).

The “bridge” between Property \( Z \) in \( s \) and Property \( Z \) in \( I^\infty \) is given by the statement, proved in [4], that for any closed set \( K \) in \( s \) with Property \( Z \) in \( s \), \( \text{Cl } K \) in \( I^\infty \) has Property \( Z \) in \( I^\infty \).

**Outline of the Proof of Theorem II.** Since an endslice in \( I^\infty \) has Property \( Z \) in \( I^\infty \), \( B(I^\infty) \) is a countable union of closed sets with Property \( Z \) in \( I^\infty \). Hence necessity follows immediately. We shall reduce the proof of sufficiency to three elementary but nontrivial theorems whose formulations require some additional definitions.

A core is a set \( C = \prod_{j>0} J_j \) where for each \( j>0, J_j \) is a closed interval contained in \( I_j^\circ \). A basic core set \( M \) structured on a core \( C = \prod_{j>0} J_j \) is defined as \( M = \{ (x_j)_{j>0} \in s \mid \text{for all but finitely many } j, x_j \in J_j \} \). A core set is a subset of \( s \) which is \( \sigma \)-compact and contains a basic core set. It is easy to verify that a basic core set is a core set.

**Theorem III.** Every core set is an apparent boundary of \( I^\infty \).
Theorem IV. For any basic core set $M$ there is a homeomorphism $g$ of $I^\infty$ onto $I^\infty$ such that $g(M) = B(I^\infty)$, and $g \circ g$ is the identity.

Theorem V. For any set $K \subseteq I^\infty$ which is the countable union of closed sets with Property Z in $I^\infty$, there exist a homeomorphism $f$ of $I^\infty$ onto $I^\infty$ and a basic core set $M$ such that $f(K) \cap M = \emptyset$, and $f(B(I^\infty)) = B(I^\infty)$.

Theorems III and IV can be proved by a more delicate argument than that outlined in [4] for the proof of Theorem 11.1 there, together with selected apparatus like that found in [2]. Theorem V can be proved rather routinely from Lemma 6.1 of [4]. We now give a short proof of sufficiency for Theorem II based on Theorems III, IV, and V.

Proof of Sufficiency for Theorem II. Let $K$ be as in the hypothesis. Let $f$ be as in Theorem V, and $g$ be as in Theorem IV. Let, by Theorem III, $h$ carry $g \circ f(K)$ onto $B(I^\infty)$. Then $h \circ g \circ f$ is the desired homeomorphism.

Bibliography


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