STRONGLY NEGLIGIBLE SETS IN FRÉCHET MANIFOLDS

BY R. D. ANDERSON[1]

Communicated August 9, 1968

Let s denote the linear metric space which is the countable infinite product of lines. It is known [1] that s is homeomorphic to Hilbert space l_2 and, in light of [8] and [10], to all separable infinite-dimensional Fréchet spaces (and therefore, of course, to all such Banach spaces). We define a Fréchet *manifold* or *F-manifold* to be a separable metric space which admits an open cover by sets homeomorphic to open subsets of s. Banach manifolds, which may be similarly defined, have been studied by a number of authors. From the results cited above it follows that all separable metric Banach manifolds modeled on separable infinite-dimensional Banach spaces are, in fact, F-manifolds. Also, clearly, any open subset of an F-manifold is an F-manifold.

In this paper, we are concerned with homeomorphisms of F-manifolds onto dense subsets of themselves. The first result of the type we consider was due to Klee [11], who showed that for any compact set K in l_2, l_2 is homeomorphic to $l_2 \backslash K$. Recently, there have been a number of results [2], [3], [4], [5], [7], [13], etc., showing that for various types of subsets K of certain linear metric spaces X, X is homeomorphic to $X \backslash K$. Bessaga [7] introduced the term "negligible" for such sets K. In some cases K was assumed compact, in others σ-compact (i.e. the countable union of compact sets) and in others K was assumed to be the countable union of closed sets of infinite deficiency (i.e. of infinite codimension). Indeed several different geometric methods [2], [3], [5], [7], [11] have been used to establish negligibility in various spaces. The results that σ-compact subsets of l_2 and of s are negligible were used in the proofs [1] and [5] that l_2 is homeomorphic to s. Questions of negligibility of subsets in Fréchet and Banach manifolds have also arisen. Where differentiable structures are assumed as for Banach spaces and manifolds and K is assumed closed, Bessaga [7], Corson, Eells and Kuiper [9], Kuiper and Burghelea [12], Moulis [13], Renz [15] and West have investigated conditions under which X and $X \backslash K$ are diffeomorphic,

[1] This research was supported in part under NSF Grant GP 6867. A manuscript giving detailed arguments for Theorems I to V is in preparation.

or they have used results of this type in other work. However, the results being announced in this paper are concerned only with homeomorphisms, not with diffeomorphisms.

In [**6**], Henderson, West, and the author introduced the concept of strong negligibility and characterized the strongly negligible closed subsets of an F-manifold. A subset K of a space X is *strongly negligible* if for any open cover G of X there exists a homeomorphism h of X onto $X\backslash K$ such that h is limited by G, i.e., for any $p\in X$ there exists $g\in G$ such that both p and $h(p)$ are elements of g.

A similar concept related to the metric of a space is the concept of metric negligibility. A set K in a metric space X is *metrically negligible* in X if for each $\epsilon>0$, there exists a homeomorphism h of X onto $X\backslash K$ such that h moves no point more than ϵ. Clearly, in a metric space X, strong negligibility of a set K implies metric negligibility since we may select an open cover of X of mesh less than ϵ. It is nontrivial, but follows from Theorem I below that, in an F-manifold, metric negligibility of a set K implies strong negligibility of K.

Following [**4**], a closed set K has *Property Z* in a space X if for each nonnull homotopically trivial open set U in X, $U\backslash K$ is nonnull and homotopically trivial. (A set U is homotopically trivial if every map of an n-sphere S^n, $n\geqq 0$, into U can be extended to a map into U of an $(n+1)$-ball bounded by S^n.) In a sense, Property Z is "trivial homotopy negligibility." See [**9**] for a similar point-of-view.

The following theorem is proved in [**6**].

THEOREM 0. *A closed set K in an F-manifold X is strongly negligible iff K has Property Z.*

It should be noted that every compact set in an F-manifold X has Property Z in X, that every closed set of infinite deficiency in s or in a separable metric Banach space has Property Z in such space, and that every closed set which is a countable union of closed sets with Property Z in an F-manifold X has Property Z in X.

The principal result of this paper is the following theorem.

THEOREM I. *A set K in an F-manifold X is strongly negligible (or metrically negligible) in X iff K is a countable union of closed sets with Property Z in X.*

Theorem I includes, as special cases or easy corollaries, Theorem 0 and many or all of the previous results on negligibility in F-manifolds X under homeomorphisms of X onto dense subsets of itself.

The proof of necessity in Theorem I is fairly straightforward. We do not outline it here.

The proof of sufficiency depends heavily on the canonical compactification of s as the Hilbert cube I^∞ in which s is regarded as a product of open intervals and the Hilbert cube is regarded as the product of the closures of the open intervals. Thus $I^\infty = \prod_{j>0} I_j$ and $s = \prod_{j>0} I_j^0$ where for each $j>0$, $I_j = [-1, 1]$ and $I_j^0 = (-1, 1)$. We let $B(I^\infty)$ denote $I^\infty \backslash s$. A set $K \subset I^\infty$ is an *apparent boundary* of I^∞ if there exists a homeomorphism h of I^∞ onto I^∞ such that $h(K) = B(I^\infty)$.

In [6], a rather general procedure for reducing certain homeomorphism problems on F-manifolds to homeomorphism problems on the Hilbert cube or on s itself is given. The actual homeomorphism theorems on I^∞ and s that are needed in [6] can be found in [2], [4], [5]. While we use the general procedures of [6] (with slight modifications) to establish sufficiency in Theorem I, we also use the following new homeomorphism theorem about I^∞.

THEOREM II. *Let $I^\infty \supset K \supset B(I^\infty)$. Then K is an apparent boundary of I^∞ iff K is a countable union of closed sets with Property Z in I^∞.*

In effect, Theorem II characterizes those apparent boundaries of I^∞ which contain $B(I^\infty)$.

The sufficiency statement of Theorem II can be used to prove the somewhat stronger Theorem IIA below, which is in a form more readily adaptable for application to F-manifolds. An *endslice* of I^∞ is a set W such that for some $i>0$, $W = \{(x_j) \in I^\infty \mid x_i = 1 \text{ (or } -1)\}$.

THEOREM IIA. *Let W^* be a finite union of endslices in I^∞, let $\epsilon > 0$, and let K be a countable union of closed sets with Property Z in I^∞ such that $K \cap W^* = \emptyset$. Then there exists a homeomorphism h of I^∞ onto I^∞ such that $h \mid W^* =$ identity, $h(s \backslash K) = s$, and h moves no point more than ϵ.*

The "bridge" between Property Z in s and Property Z in I^∞ is given by the statement, proved in [4], that for any closed set K in s with Property Z in s, Cl K in I^∞ has Property Z in I^∞.

OUTLINE OF THE PROOF OF THEOREM II. Since an endslice in I^∞ has Property Z in I^∞, $B(I^\infty)$ is a countable union of closed sets with Property Z in I^∞. Hence necessity follows immediately. We shall reduce the proof of sufficiency to three elementary but nontrivial theorems whose formulations require some additional definitions.

A *core* is a set $C = \prod_{j>0} J_j$ where for each $j>0$, J_j is a closed interval contained in I_j^0. A *basic core set* M structured on a core $C = \prod_{j>0} J_j$ is defined as $M = \{(x_j)_{j>0} \in s \mid$ for all but finitely many j, $x_j \in J_j\}$. A *core set* is a subset of s which is σ-compact and contains a basic core set. It is easy to verify that a basic core set is a core set.

THEOREM III. *Every core set is an apparent boundary of I^∞.*

THEOREM IV. *For any basic core set M there is a homeomorphism g of I^∞ onto I^∞ such that $g(M)=B(I^\infty)$, and $g \circ g$ is the identity.*

THEOREM V. *For any set $K \subset I^\infty$ which is the countable union of closed sets with Property Z in I^∞, there exist a homeomorphism f of I^∞ onto I^∞ and a basic core set M such that $f(K) \cap M = \varnothing$, and $f(B(I^\infty)) = B(I^\infty)$.*

Theorems III and IV can be proved by a more delicate argument than that outlined in [4] for the proof of Theorem 11.1 there, together with selected apparatus like that found in [2]. Theorem V can be proved rather routinely from Lemma 6.1 of [4]. We now give a short proof of sufficiency for Theorem II based on Theorems III, IV, and V.

PROOF OF SUFFICIENCY FOR THEOREM II. Let K be as in the hypothesis. Let f be as in Theorem V, and g be as in Theorem IV. Let, by Theorem III, h carry $g \circ f(K)$ onto $B(I^\infty)$. Then $h \circ g \circ f$ is the desired homeomorphism.

BIBLIOGRAPHY

1. R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **72** (1966), 515–519.

2. ———, *Topological properties of the Hilbert cube and the infinite product of open intervals*, Trans. Amer. Math. Soc. **126** (1967), 200–216.

3. ———, *On a theorem of Klee*, Proc. Amer. Math. Soc. **17** (1966), 1401–1404.

4. ———, *On topological infinite deficiency*, Michigan Math. J. **14** (1967), 365–383.

5. R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792.

6. R. D. Anderson, David W. Henderson and James E. West, *Negligible subsets of infinite-dimensional manifolds*, Compositio Math. (to appear).

7. C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 27–31.

8. C. Bessaga and A. Pelczynski, *Some remarks on homeomorphisms of F-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 265–270.

9. J. Eells and N. Kuiper, *Homotopy negligible subsets of infinite-dimensional manifolds*, Compositio Math. (to appear).

10. M. I. Kadec, *On topological equivalence of separable Banach spaces*, Dokl. Akad. Nauk SSSR **167** (1966), 23–25 = Soviet Math Dokl. **7** (1966), 319–322.

11. V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. **74** (1953), 10–43.

12. Nicolaas H. Kuiper and Dan Burghelea, *Hilbet manifolds*, preprint.

13. N. Moulis, *Sur les variétés Hilbertiennes et les fonctions non-dégénerées*, preprint.

14. Bor-Luh Lin, *Two topological properties of normed linear spaces*, Trans. Amer. Math. Soc. **114** (1965), 156–175.

15. Peter Renz, *Smooth extensions and extractions in infinite-dimensional Banach spaces*, Dissertation, University of Washington, Seattle, Wash., 1968.

LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803