IDEALS IN GROUP ALGEBRAS

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Throughout this note G denotes a locally compact abelian group and a Hausdorff space. The ideal structure of the group algebra $L_1(G)$ is still not fully known. For example, at a recent international symposium on functional analysis held at Sopot, Poland, the following questions were asked: (i) find maximal nonclosed ideals in $L_1(G)$ and (ii) what type of prime ideals are in $L_1(G)$? The following theorems answer these questions.

THEOREM 1. *Every maximal ideal of G is regular and, therefore, closed.*

In view of Theorem 1 we have the following

LEMMA 1. *If I is an ideal in $L_1(G)$ such that I is contained in exactly one maximal ideal, say M, then $\bar{I}=M$ ($\bar{I}$ is the closure of I).*

LEMMA 2. *If a prime ideal I of $L_1(G)$ is contained in a maximal ideal, then I is contained in only one maximal ideal.*

LEMMA 3. *If I is an ideal of $L_1(G)$ such that I is contained in no maximal ideal, then $\bar{I}=L_1(G)$.*

LEMMA 4. *Suppose I is a prime ideal of $L_1(G)$ such that I is contained in no maximal ideal and M is a maximal ideal in $L_1(G)$. If $J=I\cap M$, then $\bar{J}=M$ (this holds for every M).*

THEOREM 2. *If I is a prime ideal in $L_1(G)$, then I is maximal if and only if I is closed.*

Theorems 1 and 2 stated above answer questions raised at the Sopot symposium. In what follows $\hat{G}$ denotes the dual group of G.

THEOREM 3. *If $\hat{G}$ contains an infinite set, then $L_1(G)$ contains nonclosed prime ideals.*

(By the previous theorem each one is nonmaximal. The converse is true. See Corollary 4 below.)

THEOREM 4. *The following two statements are equivalent.*

(1) *Each prime ideal is contained in a unique maximal ideal.*

(2) *G is a discrete group.*

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COROLLARY 4. *Each prime ideal in $L_1(G)$ is a maximal ideal if and only if G is a finite group.*

(This makes Theorem 3 an if and only if proposition.)

Hence we have three possible types of prime ideals in $L_1(G)$: (1) closed prime ideals; i.e., the maximal regular ideals, (2) nonclosed prime ideals having the property that each such ideal is contained in a unique maximal ideal, and (3) nonclosed prime ideals having the property that each such ideal is dense in $L_1(G)$; i.e., contained in no maximal ideal. In finite groups only the first type exists. In infinite discrete groups the first two types coexist. In infinite nondiscrete groups all three types coexist.

We shall now indicate the proof. The crux of the matter is Theorem 1. It is proved in two steps. First recall that a commutative Banach algebra R has a nonregular maximal ideal if and only if $R^2 \subsetneqq R$ (cf. [**3**, pp. 87–88, 96]). The second step follows from a theorem by N. T. Varopoulos [**5**] which asserts that if R has continuous involution and a uniformly bounded set for an approximate identity, then every positive functional on R is continuous. We could substitute in the second step a theorem of E. Hewitt [**2**] which shows $(L_1(G))^2 = L_1(G)$.

In view of Theorem 1, Lemmas 1, 2, 3, and 4 and Theorem 2 follow from standard results in spectral synthesis (cf. [**4**, p. 185]). Theorems 3 and 4 depend upon constructing an appropriate ideal I_0 and a multiplicative system $H \subset L_1(G)$ such that $H \cap I_0 = \emptyset$. Under these conditions there exists a prime ideal I such that $I_0 \subset I$ and $I \cap H = \emptyset$ (cf. [**1**, p. 6]). We shall present the details in another paper.

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