CURVATURE STRUCTURES AND CONFORMAL TRANSFORMATIONS

BY RAVINDRA S. KULKARNI[1]

Communicated by O. Zariski, August 9, 1968

1. The notion of a "curvature structure" was introduced in §8, Chapter 1 of [1]. In this note we shall consider some of its applications. The details will be presented elsewhere.

Let (M, g) be a Riemann manifold. Whenever convenient, we shall denote the inner product defined by g, by $\langle\ \rangle$.

DEFINITION. A curvature structure on (M, g) is a (1, 3) tensor field T such that, for any vector fields X, Y, Z, W on M,

$$T(X, Y) = -T(Y, X) \tag{1}$$

$$\langle T(X, Y)Z, W\rangle = \langle T(Z, W)X, Y\rangle \tag{2}$$

$$T(X, Y)Z + T(Y, Z)X + T(Z, X)Y = 0. \tag{3}$$

Such a curvature structure naturally defines the corresponding "sectional curvature" K_T which is a real valued function on $G_2(M)$, the Grassmann bundle of 2-planes on M; namely, for $x \in M$, $\sigma = \{X, Y\}$ a 2-plane at x,

$$K_T(\sigma) = \frac{\langle T(X, Y)X, Y\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}.$$

As the following results show, these sectional curvature functions are of considerable geometric interest.

2. Examples of curvature structures.

(a) *A trivial curvature structure.* Consider the (1, 3) tensor field I given by

$$I(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X.$$

In this case, $K_I \equiv$ constant.

(b) *Riemann curvature structure.* This is the usual curvature structure defined by the metric g; namely, if ∇ denotes the corresponding covariant derivative,

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

[1] This research was supported by National Science Foundation grant GP-6585.

We shall denote the corresponding sectional curvature K_R simply by K.

Call two Riemann manifolds (M, g), $(\overline{M}, \bar{g})$ *"isocurved"* if there exists a sectional curvature-preserving diffeomorphism, i.e., there exists a diffeomorphism $f: M \to \overline{M}$ such that for every $x \in M$, for every $\sigma \in G_2(M)_x$, $K(\sigma) = \overline{K}(f_* \sigma)$. ($K$, resp. $\overline{K}$, are sectional curvatures canonically defined by g, resp. $\bar{g}$.)

In [2] and [3] we have shown the following converse of the "theorema egregium."

THEOREM 1. *Suppose that (M, g), $(\overline{M}, \bar{g})$ are isocurved,* dim $M \geqq 4$, *g analytic and $K \not\equiv$*constant. *Then (M, g), $(\overline{M}, \bar{g})$ are isometric.*

The methods developed in the proof of Theorem 1 are used in the following.

(c) *Ricci curvature structure.* Recall that the Riemann curvature tensor R defines the Ricci tensor via — for $x \in M$ and $X, Y \in T_x(M)$, the tangent space at x,

$$\mathrm{Ric}(X, Y) = \text{trace}: Z \to R(X, Z)Y.$$

We shall denote by $\mathrm{Ric}_\circ$, the corresponding linear transformation defined by $\langle \mathrm{Ric}_\circ X, Y\rangle = \mathrm{Ric}\,(X, Y)$.

Consider the following tensor:

$$\begin{aligned}&\mathcal{R}\mathrm{ic}(X, Y)Z \\ &\quad = \{\mathrm{Ric}(X, Z)Y - \mathrm{Ric}(Y, Z)X + \langle X, Z\rangle \mathrm{Ric}_0\, Y - \langle Y, Z\rangle \mathrm{Ric}_0\, X\}.\end{aligned}$$

This defines a curvature structure which we shall call the Ricci curvature structure. It is easily seen that for a 2-plane σ,

$$K_{\mathcal{R}\mathrm{ic}}(\sigma) = \text{trace Ric}\,|_\sigma.$$

It is also evident that if dim $M \geqq 3$, then $K_{\mathcal{R}\mathrm{ic}} =$ constant if and only if (M, g) is an Einstein manifold (i.e., Ric $(X, Y) = \alpha\langle X, Y\rangle$ for some constant α).

Call two manifolds (M, g), $(\overline{M}, \bar{g})$ *"iso-Ricci-curved"* if there exists a $K_{\mathcal{R}\mathrm{ic}}$-preserving diffeomorphism $f: M \to \overline{M}$. We have the following

THEOREM 2. *Suppose that (M, g), $(\overline{M}, \bar{g})$ are iso-Ricci-curved,* dim $M \geqq 3$, *g analytic and $K_{\mathcal{R}\mathrm{ic}} \not\equiv$*constant. *Then (M, g), $(\overline{M}, \bar{g})$ are conformal* (i.e., *$g = h \cdot f^* g$, where h is a positive real valued function on M*).

As yet we have not been able to replace "conformal" by "isometric," except under further hypotheses.

(d) *Conformal curvature structure.* Consider the tensor field defined by

$$C = R - \frac{1}{n-2}\,\mathcal{R}\mathrm{ic} + \frac{\mathrm{Sc}}{(n-1)(n-2)}\,I.$$

(Here, $n = \dim M$, and R, $\mathcal{R}$ic, I as defined above, and Sc = scalar curvature = trace Ric_o). This tensor was first written down by Weyl. We shall call K_C, the "conformal curvature" and denote it by K_{con}.

A manifold (M, g) is called conformally flat, if locally we can write $g = h \cdot g_0$ where g_0 = Euclidean metric, and h, a positive real valued function on M. A well-known theorem of Weyl is that: if $\dim M \geqq 4$, then (M, g) is conformally flat if and only if $C = 0$. Using this theorem, we can prove

THEOREM 3. *Let (M, g) be a Riemann manifold of* $\dim \geqq 4$. *Then the following conditions are equivalent*:

(1) *(M, g) is conformally flat,*

(2) $K_{\mathrm{con}} \equiv 0$,

(3) $K_{\mathrm{con}} \equiv$ constant,

(4) *for every orthogonal 4-tuple of tangent vectors* $\{e_1, e_2, e_3, e_4\}$,

$$K(e_1, e_2), + K(e_3, e_4), = K(e_1, e_4) + K(e_2, e_3).$$

Note that (4) is a characterization of a conformally flat space purely in terms of sectional curvature.

Call two Riemann manifolds (M, g), $(\overline{M}, \bar{g})$ *"isoconformally curved"* if there exists a K_{con}-preserving diffeomorphism among them. We have

THEOREM 4. *Let (M, g), $(\overline{M}, \bar{g})$ be isoconformally curved, g analytic,* $\dim M \geqq 4$, *and* $K_{\mathrm{con}} \not\equiv$ constant. *Then (M, g), $(\overline{M}, \bar{g})$ are isometric.*

3. **Conformal transformations.** Consideration of K_{con} leads to some interesting results about conformal maps of Riemann manifolds. For convenience, we shall restrict to conformal maps of a Riemann manifold onto itself. A natural question is: when does (M, g) admit nontrivial conformal maps?

In this direction, a classical theorem of Liouville says that every conformal map of the Euclidean space $\boldsymbol{R}^n$, $n \geqq 3$, with the standard metric, is either an isometry or a homothety.

A significant partial generalization of this theorem was obtained by Yano and Nagano [4]: a complete Einstein space of $\dim \geqq 3$, admitting a 1-parameter group of nonhomothetic, conformal transformations is compact and in fact isometric with a standard sphere.

We have been able to generalize this theorem by weakening the hypothesis, where "1-parameter group of nonhomothetic conformal transformations" is replaced by a "*single* nonhomothetic conformal transformation." Moreover we have shown that even "completeness" (at least *generically*) is not necessary. A typical result is the following.

THEOREM 5. *Let* (M, g) *be an Einstein manifold of* dim $\leqq 4$, g *analytic and* $K \not\equiv$ constant. *Then every conformal map of* (M, g) *onto itself is a homothety.*

REMARKS. (1) In the above situation a conformal map is in fact an isometry if $\mathrm{Sc} \not\equiv 0$.

(2) The local results (like Theorem 5) do not use positive definiteness of the metric. In particular Theorem 5 applies to the space of general relativity where the energy momentum tensor vanishes.

(3) Theorem 5 also holds if we replace the hypothesis "dim $M \leqq 4$," by "dim $M \geqq 5$" and a *generic* hypothesis about K, e.g., the set

$$\{x \in M \mid K|_{G_2(M)_x} \text{ has only nondegenerate critical points}\}$$

is dense in M.

Various results which were based on the result of Yano and Nagano—(e.g., an important result due to Goldberg and Kobayashi [5]), and also the results with a different flavor depending on the sign of sectional curvature—(e.g., Lichnerowicz [6, §83]) can also be improved in a similar way.

REFERENCES

1. R. S. Kulkarni, *Curvature and metric*, Ph.D. Thesis, Harvard University, Cambridge, Mass., 1967.

2 and 3. ———, *Une réciproque du "theorema egregium,"* C. R. Acad. Sci. Paris **A264** (1967), 944–946; **A265** (1967), 388–390.

4. K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformation*, Ann. of Math. (2) **69** (1959), 451–461.

5. S. Goldberg and S. Kobayashi, *The conformal transformation group of a compact homogeneous Riemann manifold*, Bull. Amer. Math. Soc. **68** (1962), 378–381.

6. A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, Paris, 1958.

HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138