DIFFERENTIABLE FUNCTIONS ON c_0

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If E and F are two Banach spaces, denote by $C^{p,q}(E, F)$, $0 \leqq q \leqq p \leqq \infty$, those functions in $C^p(E, F)$ whose derivatives of order less than or equal to q are bounded. Call a Banach space, E, $C^{p,q}$ smooth if there exists a nonzero $C^{p,q}$ function on E with bounded support. Then finite dimensional spaces are $C^{\infty,\infty}$ smooth and if an L_p space is C^q smooth it is also $C^{q,q}$ smooth. Although c_0 is known to possess a C^∞ (away from zero) norm as described in Bonic and Frampton [1], it is a consequence of the following theorem that c_0 is not $C^{2,2}$ smooth.

THEOREM. *Let $f \in C^1(c_0, R)$ with Df uniformly continuous. Then the support of f is unbounded.*

PROOF. If not then there would exist an $f \in C^1(c_0, R)$ such that $f(0)=1$, $f(x)=0$ for $\|x\| \geqq 1$ and Df is uniformly continuous. Pick N such that $\|h\| \leqq 1/N$ implies $\|Df(x+h)-Df(x)\| \leqq 1/2$. Then the mean value theorem gives that $|f(x+h)-f(x)-Df(x)(h)| \leqq 1/2\|h\|$ when $\|h\| \leqq 1/N$. Let A be the set of all x in c_0 such that 2^N-1 of the first 2^N components of x have absolute value $1/N$, the remaining component has absolute value less than or equal to $1/N$ and all the components after the first 2^N are zero. Since A is connected and even, we can pick inductively $h_1, \cdots h_N \in A$ such that $Df(h_1 + \cdots + h_{k-1}) \cdot (h_k) = 0$ and $h_1 + \cdots + h_k$ has at least 2^{n-k} components equal to k/N. Then

$$\|h_1 + \cdots + h_N\| = 1$$

and

$$\begin{aligned} |f(h_1 + \cdots + h_N) - f(0)| &\leqq \sum_{k=1}^{N} |f(h_1 + \cdots + h_k) - f(h_1 + \cdots + h_{k-1}) \\ &\quad - Df(h_1 + \cdots + h_{k-1})h_k| \leqq \sum_{k=1}^{N} \frac{1}{2}\|h_k\| = \frac{1}{2} \end{aligned}$$

which is a contradiction.

COROLLARY 1. *Let $f \in C^1(c_0, R)$ and Df be uniformly continuous. Then $f(\delta U)$ is dense in $f(U)$ for all bounded open sets U.*

COROLLARY 2. *There exists a closed subset of c_0 which is not the loci of zeros of a C^2 function.*

PROOF. Consider the complement of a sequence of disjoint open balls converging to a point.

The C^∞ norm described in [1] has first derivative bounded by one and by composing with a suitable function in $C^\infty(R, R)$ we get a $C^{\infty,1}$ function on c_0 with bounded support. The following is another example of a $C^{\infty,1}$ function.

Let $\eta \in C^\infty(R, R)$, $\eta(t) \geqq 0$, $\eta(t) = 0$ if $|t| \geqq 1/4$ and $\int_{-1/4}^{1/4} \eta(t)dt = 1$. Define

$$\phi_n(x) = \int_{-1/4}^{1/4} \cdots \int_{-1/4}^{1/4} \eta(y_1) \cdots \eta(y_n) \cdot F(x_1 + y_1, \cdots, x_n + y_n, x_{n+1}, \cdots) dy_1 \cdots dy_n$$

where $F(x) = \inf_{\|y\| \leq 1} (\|x - y\|)$, $x = \{x_1, x_2, \cdots\}$, $y = \{y_1, y_2, \cdots\}$. Suppose that $|x_m| \leqq 1/4$ if $m > n(x)$. Now if $\|x' - x\| \leqq 1/4$, $\|y\| \leqq 1/4$ and $x'_m = x_m$ for $m \leqq n(x)$, then $F(x' + y) = F(x + y)$. Hence when $\|z - x\| \leqq 1/4$, $\phi_{n(x)}(z)$ depends only on the first $n(x)$ coordinates and therefore is C^∞. Also $\|x' - x\| \leqq 1/4$, $\|y\| \leqq 1/4$ and $y_1 = \cdots = y_{n(x)} = 0$ imply that $F(x' + y) = F(x')$. Hence $\phi_m(x') = \phi_{n(x)}(x')$ when $m \geqq n(x)$ and $\|x' - x\| \leqq 1/4$. The above implies that $\phi(x) = \lim_{n \to \infty} \phi_n(x)$ exists and is C^∞ for all x. Now $|\phi_n(x) - \phi_n(z)| \leqq \int_{-1/4}^{1/4} \cdots \int_{-1/4}^{1/4} \eta(y_1) \cdots \eta(y_n) \|x - z\| dy_1 \cdots dy_n = \|x - z\|$. Hence $|\phi(x) - \phi(z)| \leqq \|x - z\|$ which gives $\|D\phi(x)\| \leqq 1$ for all x. Finally let $\rho \in C^\infty(R, R)$, $0 \leqq \rho \leqq 1$, $\rho(t) = 1$ if $t \leqq 0$ and $\rho(t) = 0$ if $t \geqq 3/4$. Then $\rho(\phi(x)) \in C^{\infty,1}(c_0, R)$, $\rho(\phi(0)) = 1$ and the support of $\rho(\phi(x))$ is contained in the unit ball.

BIBLIOGRAPHY

1. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. **15** (1966), 877–898.

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