ASYMPTOTIC PROPERTIES OF ENTIRE FUNCTIONS
EXTREMAL FOR THE \( \cos \pi \rho \) THEOREM

BY DAVID DRASIN AND DANIEL F. SHEA\textsuperscript{1}

Communicated by W. Fuchs, September 6, 1968

Let \( f(z) \) be an entire function of order \( \rho < 1 \). The classical "\( \cos \pi \rho \) theorem" of Valiron and Wiman [4, pp. 40, 51] asserts that if

\[
\mu(r) = \min_{|z|=r} |f(z)|, \quad M(r) = \max_{|z|=r} |f(z)|,
\]

then, given \( \epsilon > 0 \), the inequality

\[
(1) \quad \log \mu(r) > (\cos \pi \rho - \epsilon) \log M(r)
\]

holds for a sequence \( r = r_n \to +\infty \).

We consider those functions \( f(z) \) for which (1) is the best possible inequality, and discuss the global asymptotic behavior of such functions.

**Theorem 1.** Let \( f(z) \) be an entire function of order \( \rho \) \((0 \leq \rho < 1)\), and suppose

\[
(2) \quad \log \mu(r) \leq [\cos \pi \rho + \epsilon(r)] \log M(r)
\]

where \( \epsilon(r) \to 0 \) as \( r \to \infty \).

Then there exists a set \( E \) of logarithmic density zero and a slowly varying function\textsuperscript{2} \( \psi(r) \) such that

\[
(3) \quad \log M(r) = r \psi(r) \quad (r \in E),
\]

\[
(4) \quad n(r, 0) = [\sin \pi \rho /\pi + o(1)] r \psi(r) \quad (r \to \infty, r \in E)
\]

(\text{where, as usual,} \( n(r, 0) \) \text{denotes the number of zeros of} \( f(z) \) \text{in} \(|z| \leq r\)),

\[
(5) \quad \log \mu(r) = [\cos \pi \rho + o(1)] r \psi(r) \quad (r \to \infty, r \in E \cup H),
\]

where \( H \) has (linear) density zero.

Further, there exists a real-valued function \( \theta(r) \) such that if \( k > 1 \) and \( \delta > 0 \) are given and \( n(r) \) denotes the number of zeros of \( f(z) \) in the region

\textsuperscript{1} The first author was partially supported by NSF grant 4192-50-1395; the second author was partially supported by NSF grant GP-5728.

\textsuperscript{2} A function \( \psi(r) \) is said to vary slowly if it is defined and positive for all \( r > r_0 \) and satisfies \( \lim_{r \to \infty} \psi(\sigma r) /\psi(r) = 1 \) \((0 < \sigma < \infty)\). For a useful discussion of the properties of such functions see, for example, [9, p. 419]. For a discussion of linear and logarithmic densities see [4, p. 5].
\[ \{ z : k^{-1}r \leq |z| \leq kr, \delta \leq |\arg z - \theta(r)| \leq \pi \}, \]

then

\[ \nu(r) = o(r^\psi(r)) \quad (r \to \infty, r \not\in E). \]

The function \( \theta(r) \) oscillates slowly outside of \( E \), in the sense that if \( k>1 \) and \( \epsilon>0 \) are given, then

\[ |\theta(t) - \theta(r)| < \epsilon \quad (r > r_0(\epsilon, k), r \not\in E) \]

holds for all \( t \) in the interval \( k^{-1}r \leq t \leq kr \).

The content of the conclusions (3)–(7) can be expressed more intuitively if we say that on almost all long intervals, \( f(z) \) behaves like a Lindelöf function of order \( \rho \) [12, p. 18]. Indeed, it is not difficult to see that Theorem 1 implies that the asymptotic expansion

\[ \log |f(te^{i(\phi+\psi)})| = [\cos \phi + o(1)]\psi(r)r^\rho \]

\[ (k^{-1}r \leq t \leq kr, t \not\in H, |\phi| \leq \pi), \]

is valid (uniformly in \( t \) and \( \phi \)) as \( r \to \infty \) outside \( E \), where \( \phi_0 = \theta(r) - \pi \), \( k>1 \) is a given constant, and \( H \) is the set of density zero given in Theorem 1.

Recent examples\(^4\) of W. K. Hayman [10] show that some exceptional set \( E \) must be present in Theorem 1; when coupled with Theorem 2 below, they also show that even in the important special case when all the zeros of \( f(z) \) are negative, \( E \) cannot be replaced by a set of linear density 0.

Theorem 1 may be compared to recent results of Kjellberg [11], Essén [7], Essén-Ganelius [8], and Anderson [1]. These authors consider (2) from another point of view; in particular, \( \rho \) can be any number, \( 0<\rho<1 \) (not necessarily the order of \( f(z) \)), but on the other hand, \( \epsilon(r) \) must satisfy some condition such as

\[ \limsup \int_{R_1}^{R_2} \frac{e(r) \log M(r)}{r^{1+\rho}} \, dr < M < \infty. \]

Their conclusion, that \( \log M(r)/r^\rho \) tends to a limit \( \alpha \) (\( 0 \leq \alpha \leq \infty \)) as \( r \to \infty \) (with no need to avoid an exceptional set), is also of a different nature than that deduced here.

1. Outline of the proof. Let \( f(z) \) satisfy the hypotheses of the theorem. We can assume that \( f(0) = 1 \), and write

\(^4\) Hayman's examples are valid only if \( \rho = 1/2 \), but he comments that this case is probably typical.
1969] ASYMPTOTIC PROPERTIES FOR THE \cos \pi \rho THEOREM 121

\[1.1\] \quad f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right), \quad F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{|a_n|}\right).

Since

\[1.2\] \quad \log \left| F(-r) \right| + \log F(r) \leq \log \mu(r) + \log M(r)

[4, p. 40], it follows at once that \( F(z) \) also satisfies the hypotheses of Theorem 1, and hence a theorem of P. D. Barry [3] yields that

\[ \log \left| F(-r) \right| = \left[ \cos \pi \rho + o(1) \right] \log F(r) \quad (r \to \infty, r \in G^*), \]

where \( G^* \) has logarithmic density one. It is not hard to see, using (1.2) and hypothesis (2) again, that

\[1.3\] \quad \log M(r) \sim \log F(r) \quad (r \to \infty, r \in G^*).

An easy extension of Theorem 2 of [2] now shows that from (1.3) follows

\[1.4\] \quad \nu(r) = o(\log M(\rho)) \quad (r \to \infty, r \in G^*).

We next establish that \( G^* \) can be replaced by a subset \( G_* \) having the following crucial properties: there are sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) and a set \( H \) of (linear) density zero such that

\[ G_* = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] - H \quad (\alpha_n \to \infty, \beta_n/\alpha_n \to \infty) \]

has logarithmic density one, and, if \( k > 1 \) is given, then

\[ [k^{-1}\alpha_n, k\beta_n] \subseteq G^* \cup H \quad (n > n_0(k)). \]

The exceptional set \( E \) which appears in the statement of Theorem 1 is the complement of \( G = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] \).

In view of elementary properties of sets of linear density zero, it is easy to see that (1.3) holds with \( G^* \) replaced by \( G \), and so it suffices to prove (3)–(5) for \( F(z) \). The argument hinges now on a suitable generalization (to allow exceptional sets of logarithmic density zero) of the following theorem [6], which is one form of a complement to some classical results of Titchmarsh [13] and Bowen and Macintyre [5].

**Theorem 2.** Let \( F(z) \) be an entire function of the form (1.1), and suppose

\[ \frac{\log |F(-r)|}{\log F(r)} \to \alpha \quad (r \to \infty, r \in H), \]

where \( H \) is of (linear) density zero.
Then $-1 \leq \alpha \leq 1$, and

$$\log F(r) = r^\alpha \psi(r),$$

$$n(r,0) = \left[ \frac{\sin \pi \rho}{\pi} + o(1) \right] r^\alpha \psi(r) \quad (r \to \infty),$$

$$\log |F(-r)| = [\cos \pi \rho + o(1)] r^\alpha \psi(r) \quad (r \to \infty, r \in H),$$

where $\rho$ is determined by

$$\cos \pi \rho = \alpha \quad (0 \leq \rho \leq 1),$$

and $\psi$ is a slowly varying function.

Finally, (6) follows from (1.4) and (3), and (7) is an easy consequence of (6) (cf. [2, Corollary 1]).

Conclusion (3) of Theorem 1 implies that $f(z)$ has regular growth in the sense of Valiron. Analogues of Theorem 1, valid for functions of irregular growth, can also be derived from these methods.

REFERENCES

10. W. K. Hayman, *Some examples related to the $\cos \pi \rho$-theorem* (to appear).

Purdue University, Lafayette, Indiana 47907 and University of Wisconsin, Madison, Wisconsin 53706