EXTENDING COHERENT ANALYTIC SHEAVES THROUGH SUBVARIETIES

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We announce the following

THEOREM I. *Suppose V is a subvariety of dimension $\leqq n$ in a (not necessarily reduced) complex space X and $\mathfrak{F}$ is a coherent analytic sheaf on $X-V$ with* codh $\mathfrak{F} \geqq n+3$. *Let $\theta: X-V \to X$ be the inclusion map. Then $\theta_0(\mathfrak{F})$ is a coherent analytic sheaf on X extending $\mathfrak{F}$ (where $\theta_q(\mathfrak{F})$ is the qth direct image of $\mathfrak{F}$ under θ).*

The case $n=0$ was proved in [7]. The case where X is a manifold of dimension $n+3$ was proved in [5].

We give here only a very brief outline of the proof together with some related results and application. Details will appear elsewhere.

Suppose $\mathfrak{F}$ is an analytic sheaf on a complex space X and n is a nonnegative integer. We denote by $\mathfrak{F}^{[n]}$ the analytic sheaf on X defined by the following presheaf: if $U \subset W$ are open subsets of X, then $\mathfrak{F}^{[n]}(U)=$ the direct limit of $\{\Gamma(U-A, \mathfrak{F}) \mid A \in \mathfrak{A}\}$, where $\mathfrak{A}$ is the set of all subvarieties of dimension $\leqq n$ in U directed by inclusion, and $\mathfrak{F}^{[n]}(W) \to \mathfrak{F}^{[n]}(U)$ is induced by restriction maps. There is a canonical sheaf-homomorphism from $\mathfrak{F}$ to $\mathfrak{F}^{[n]}$. We denote by $O_{[n]}\mathfrak{F}$ the analytic subsheaf of $\mathfrak{F}$ defined as follows: for $x \in X$, $s \in (O_{[n]}\mathfrak{F})_x$ if and only if there exist an open neighborhood U of x in X, a subvariety A in U of dimension $\leqq n$, and $t \in \Gamma(U, \mathfrak{F})$ such that $t_x = s$ and $t_y=0$ for $y \in U-A$.

PROPOSITION 1 [6]. *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a complex space X and n is a nonnegative integer.*

(a) *If $O_{[n+1]}\mathfrak{F}=0$, then $\mathfrak{F}^{[n]}$ is coherent and the subvariety where $\mathfrak{F}^{[n]}$ is not isomorphic to $\mathfrak{F}$ canonically is of dimension $\leqq n$.*

(b) *If $\mathfrak{F}$ is canonically isomorphic to $\mathfrak{F}^{[n]}$, then $O_{[n+1]}\mathfrak{F}=0$.*

The following can be proved from Proposition 1 and by induction on n.

PROPOSITION 2. *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a complex space X and n is a nonnegative integer such that $\mathfrak{F}$ is canonically iso-*

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morphic to $\mathfrak{F}^{[n]}$. *Then for* $-1 \leqq k < n$ *the subvariety* $\{x \in X \mid \operatorname{codh} \mathfrak{F}_x \leqq k+2\}$ *has dimension* $\leqq k$.

Let $z_1, \cdots, z_N$ and $t_1, \cdots, t_n$ be respectively coordinates of C^N and C^n. Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be respectively the structure-sheaves of C^n and $C^N \times C^n$. For $0 \leqq a < b$ and $\rho > 0$, let

$$G(b) = \{z \in C^N \mid \max(|z_1|, \cdots, |z_N|) < b\},$$
$$G(a, b) = \{z \in G(b) \mid a < |z_i| \quad \text{for some } 1 \leqq i \leqq N\},$$

and $K(\rho) = \{t \in C^n \mid \max(|t_1|, \cdots, |t_n|) < \rho\}$. $K = K(1)$. Let $\pi: G(a, b) \times K \to K$ be the projection.

PROPOSITION 3. *Suppose* $0 < \tilde{a} < a < b < \tilde{b}$ *and* $\mathfrak{F}$ *is a coherent analytic sheaf on* $G(\tilde{a}, \tilde{b}) \times K$ *such that* $\operatorname{codh} \mathfrak{F} \geqq n+3$ *and* t_n *is not a zero-divisor for* $\mathfrak{F}_x$ *for* $x \in G(\tilde{a}, \tilde{b}) \times 0$. *Suppose* $\mathfrak{F}/t_n\mathfrak{F}$ *can be extended to a coherent analytic sheaf* $\mathfrak{G}$ *on* $G(\tilde{b}) \times K$ *such that* $\operatorname{codh} \mathfrak{G} \geqq n+1$ *and* $t_1, \cdots, t_{n-1}$ *is a* $\mathfrak{G}_x$*-sequence for* $x \in G(\tilde{b}) \times 0$. *Then* $(\pi_1(\mathfrak{F}))_0$ *is finitely generated over* $\mathcal{O}_0$.

The proof of Proposition 3 is rather complicated where modifications of techniques of [1] and [2] are used.

PROPOSITION 4. *Suppose* $a, b, \tilde{a}, \tilde{b}$, *and* $\mathfrak{F}$ *are as in Proposition* 3. *If* $z_j - z_j(x)$ *is not a zero-divisor for* $\mathfrak{F}_x$ *for* $x \in G(\tilde{a}, \tilde{b}) \times K$ *and* $1 \leqq j \leqq N$. *Then for some* $a < c < d < b$ *and* $0 < \rho < 1$ $\Gamma(G(c, d) \times K(\rho), \mathfrak{F})$ *generates* $\mathfrak{F}$ *on* $G(c, d) \times K(\rho)$.

PROOF (SKETCH). Consider

$(*)_k$ For some $a < c < d < b$ and $0 < \rho < 1$ there exists a subvariety Z in $G(c, d) \times K(\rho)$ such that $\Gamma(G(c, d) \times K(\rho), \mathfrak{F})$ generates $\mathfrak{F}$ on $G(c, d) \times K(\rho) - Z$ and $\dim Z \cap G(c, d) \times 0 \leqq k$.

The Proposition follows by proving $(*)_k$ by backward induction on k for $0 \leqq k \leqq N$. For the induction process we need only prove the following.

(†) If Z is a positive-dimensional subvariety of $G(a, b) \times 0$, then for some $x \in Z$ and some $0 < \rho < 1$ $\Gamma(G(a, b) \times K(\rho), \mathfrak{F})$ generates $\mathfrak{F}_x$.

To prove (†), choose $1 \leqq j \leqq N$ and $\{x_m\}_{m=1}^{\infty} \subset Z$ such that $|z_j(x_m)| > a$ and $|z_j(x_m)| \to b$. Let $V_m = \{x \in G(a,b) \times K \mid z_j(x) = z_j(x_m)\}$ and $V = \bigcup_{m=1}^{\infty} V_m$. Let f be a holomorphic function on $G(b) \times K$ generating the ideal-sheaf of V. The short exact sequence $0 \to \mathfrak{F} \xrightarrow{\phi} \mathfrak{F} \to \mathfrak{F}/f\mathfrak{F} \to 0$ (where ϕ is defined by multiplication by f) gives rise to the exact sequence

$$(\#) \qquad (\pi_0(\mathfrak{F}))_0 \xrightarrow{\alpha} (\pi_0(\mathfrak{F}/f\mathfrak{F}))_0 \xrightarrow{\beta} (\pi_1(\mathfrak{F}))_0.$$

Let $\gamma: \tilde{\mathcal{O}}^p \to \mathfrak{F}$ be a sheaf-epimorphism on $\{x \in G(b) \times K(\frac{1}{2}) \mid a < |z_j(x)|\}$. γ induces $\gamma': \tilde{\mathcal{O}}^p / f\tilde{\mathcal{O}}^p \to \mathfrak{F}/f\mathfrak{F}$. Let $s_m^{(i)} \in (\pi_0(\mathfrak{F}/f\mathfrak{F}))_0$ be induced under γ' by the p-tuple of holomorphic functions on V which is $(0, \cdots, 0, 1, 0, \cdots, 0)$ on V_m with 1 in the ith place and is zero otherwise. By considering the direct sum of p copies of (#) and using Proposition 3 we obtain $a_1, \cdots, a_{m-1} \in \mathcal{O}_0$ for some m such that for all $1 \leqq i \leqq p$ $\beta(s_m^{(i)} - \sum_{q=1}^{m-1} a_q s_q^{(i)}) = 0$. For some $t_m^{(1)}, \cdots, t_m^{(p)} \in (\pi_0(\mathfrak{F}))_0$, $\alpha(t_m^{(i)}) = s_m^{(i)}$. $\mathfrak{F}_{x_m}$ is generated by sections of $\mathfrak{F}$ inducing $t_m^{(1)}, \cdots, t_m^{(p)}$. Q.E.D.

PROPOSITION 5. *Suppose D is a domain in C^n, $0 \leqq a < b$, and $\mathfrak{F}$ is a coherent analytic sheaf on $G(b) \times D$. If $\mathfrak{F}^{[n-1]} \approx \mathfrak{F}$, then the restriction map $\phi: \Gamma(G(b) \times D, \mathfrak{F}) \to \Gamma(G(a, b) \times D, \mathfrak{F})$ is injective. If $\mathfrak{F}^{[n]} \approx \mathfrak{F}$, then ϕ is surjective.*

PROOF (SKETCH). The injectivity of ϕ follows from Proposition 1(b). For the surjectivity of ϕ consider first the special case codh $\mathfrak{F} \geqq n+2$. For the general case use Proposition 2 and induction on n. Q.E.D.

PROPOSITION 6. *Suppose D is a domain in C^n, $0 \leqq a < a' < b$, and $\mathfrak{F}$ is a coherent analytic sheaf on $G(a, b) \times D$ with $\mathfrak{F}^{[n+1]} \approx \mathfrak{F}$. Then the restriction map $\Gamma(G(a, b) \times D, \mathfrak{F}) \to \Gamma(G(a', b) \times D, \mathfrak{F})$ is bijective.*

PROOF (SKETCH). Use Proposition 5 and consider the restriction maps $\Gamma((G(a, b) \cap U_i) \times D, \mathfrak{F}) \to \Gamma((G(a', b) \cap U_i) \times D, \mathfrak{F})$ and $\Gamma((G(a, b) \cap U_i \cap U_j) \times D, \mathfrak{F}) \to \Gamma((G(a', b) \cap U_i \cap U_j) \times D, \mathfrak{F})$, where $U_i = \{x \in C^N \mid |z_i(x)| > a\}$. Q.E.D.

By using Propositions 1, 2, 4, 5, and 6 and by induction on n, we can obtain

THEOREM II. *Suppose D is a domain in C^N, $0 \leqq a < b$, and $\mathfrak{F}$ is a coherent analytic sheaf on $G(a, b) \times D$ with $\mathfrak{F}^{[n+1]} \approx \mathfrak{F}$.*

(a) *There exists a coherent analytic sheaf $\mathfrak{F}'$ on $G(b) \times D$ which extends $\mathfrak{F}$ and satisfies $(\mathfrak{F}')^{[n+1]} \approx \mathfrak{F}'$.*

(b) *If $\mathfrak{F}'$ and $\mathfrak{F}''$ are two coherent analytic sheaves on $G(b) \times D$ both extending $\mathfrak{F}$ such that $(\mathfrak{F}')^{[n+1]} \approx \mathfrak{F}'$ and $(\mathfrak{F}'')^{[n+1]} \approx \mathfrak{F}''$, then there exists a unique sheaf-isomorphism from $\mathfrak{F}'$ to $\mathfrak{F}''$ which is equal to the identity map of $\mathfrak{F}$ on $G(a, b) \times D$.*

As a corollary of Theorem II we have

THEOREM III. *Suppose V is a subvariety of dimension $\leqq n$ in a com-*

plex space X and $\mathfrak{F}$ is a coherent analytic sheaf on $X-V$. If $\mathfrak{F}^{[n+1]}\approx\mathfrak{F}$, then $\theta_0(\mathfrak{F})$ is a coherent analytic sheaf on X extending $\mathfrak{F}$, where $\theta: X-V \rightarrow X$ is the inclusion map.

Theorem I follows from Theorem III and Korollar zu Satz III of [**3**].

Theorem III answers in the affirmative the following question posed by Serre [**4**, p. 372]: Suppose V is a subvariety of codimension $\geqq 3$ in a normal reduced complex space X. If $\mathfrak{F}$ is a reflexive coherent analytic sheaf on $X-V$, is $\theta_0(\mathfrak{F})$ coherent (where $\theta: X-V\rightarrow X$ is the inclusion map)?

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