W-SYSTEMS ARE THE WALSH FUNCTIONS

BY DANIEL WATERMAN[1]

Communicated by Henry Helson, July 15, 1968

Let $\{\phi_n\}$, $n=0, 1, 2, \cdots$, be a system of functions on $(0, 1)$ with $\phi_0 \equiv 1$. For $n=2^{k_1}+2^{k_2}+\cdots+2^{k_s}$, with $0 \leqq k_1 < k_2 < \cdots < k_s$, we set

$$\psi_n = \phi_{k_1} \cdot \phi_{k_2} \cdot \ldots \cdot \phi_{k_s}.$$

If $\{\psi_n\}$ is an orthogonal system on $(0, 1)$, it is called a W-system [1, pp. 185–196] after the Walsh system, $\{w_n\}$, which is formed from the Rademacher functions, $\{r_n\}$, in this manner. It is generally assumed that the system $\{\psi_n\}$ is equinormed, i.e., there is a constant K such that

$$(*) \qquad \int_0^1 |\psi_n|^2 = K \qquad \text{for } n \geqq n_0.$$

The study of these systems has shown that they are essentially of two types depending on whether or not we assume

$$(**) \qquad |\phi_n(x)| \leqq 1 \text{ a.e.} \qquad \text{for every } n.$$

When this assumption is made, the results obtained on W-systems parallel those for the Walsh system. In the other case, the behavior may resemble that of systems generated by the strongly lacunary trigonometric sequences $\{2^{1/2} \cos m_n x\}$ and $\{2^{1/2} \sin m_n x\}$ with $m_{n+1}/m_n \geqq 3$ ([1, pp. 190–191] and [2, pp. 208–209]).

We will restrict our consideration to the systems satisfying (*) and (**) and we will refer to them simply as W-systems.

In very general terms our results may be stated as follows.

A result concerning the a.e. convergence or summability of a Walsh series, $\sum c_n w_n$, implies the corresponding result for the W-series, $\sum c_n \psi_n$.

We will state more precise results shortly.

From our first lemma we conclude that we may assume $|\phi_n(x)| = 1$ a.e. without loss of generality.

LEMMA 1. *To any system $\{\phi_n\}$ on $(0, 1)$ with*

$$\int |\phi_i|^2 = \int |\phi_i \phi_j|^2 = K \qquad \text{for } i \neq j$$

[1] Supported by National Science Foundation Grant GP7358.

and

$$|\phi_n(x)| \leqq 1 \text{ a.e.} \qquad \textit{for all } n,$$

there corresponds a measurable set $E \subset (0, 1)$, $m(E) = K$, *such that, for every* n, $|\phi_n| \equiv 1$ *on* E *and* $|\phi_n| = 0$ *a.e. on* E^c.

It is not difficult to see that this implies that the system $\{\psi_n\}$ lives on a set E of measure K in the sense that

(i) $|\psi_n| = 1$ on E for every n,

(ii) $m(\{\psi_n \neq 0\} \cap E^c) > 0$ for only finitely many n,

(iii) $\{\psi_n\}$ is orthogonal relative to E.

Alexits has called a system $\{\phi_n\}$ *multiplicatively orthogonal* if $\int \psi_n = 0$ for $n > 0$ and *strongly multiplicatively orthogonal* if $\{\psi_n\}$ is orthogonal [1, pp. 186–187]. We note that under the hypotheses of Lemma 1, these notions coincide relative to E.

If A is a 1-1 measure preserving map of (0, 1) onto itself taking $(0, K)$ into E, we see that $\{\psi_n \circ A(Kx)\}$ is a W-system on (0, 1) and $|\psi_n \circ A(Kx)| = 1$ a.e. for all n. Thus we can reduce the study of the original system on E to the study of a W-system living on (0, 1).

We will assume, henceforth, that $|\phi_n(x)| = 1$ a.e. for all n.

Let us consider the sets on which $\phi_1, \cdots, \phi_k$ are of constant sign. For any integer $t \in [0, 2^k - 1]$, we have a unique representation

$$t = a_k 2^0 + \cdots + a_1 2^{k-1}$$

with $a_\nu = 0$ or 1. Set

$$E_k^t = \{x : \phi_\nu(x) = e^{i\pi a_\nu}, v = 1, \cdots, k\}.$$

Then $(0, 1) = \bigcup_{t=0}^{2^k-1} E_k$, modulo a null set, and the sets E_k^t are pairwise disjoint and measurable. We have, indeed,

LEMMA 2. $m(E_k^t) = 1/2^k$ *for all* k *and* $t = 0, 1, \cdots, 2^k - 1$.

We now define a function on (0, 1) by means of the dyadic representation of its values, $y(x) = .\alpha_1\alpha_2 \cdots$, with

$$\alpha_\nu = (1/i\pi) \log \phi_\nu(x), \nu = 1, 2, 3, \cdots.$$

We see at once that, except for those points for which $\alpha_\nu \equiv 1$ or $\alpha_\nu \equiv 0$ from some index onward, $y(x) = .a_1a_2a_3 \cdots$, $a_\nu = 0$ or 1, if and only if $x \in E_k^t$, $t = 2^k(.a_1 \cdots a_k)$ for every k. This exceptional set, as well as the set for which $\alpha_\nu \neq 0$ or 1 for some ν, is easily seen to be a set of measure zero.

We have the following result.

LEMMA 3. *For every measurable* $E \subseteq (0, 1)$, $y^{-1}(E)$ *is measurable and* $m(y^{-1}(E)) = m(E)$.

A sequence $\{f_n\}$ of bounded measurable functions is said to be *maximal* if there is a set Z of measure zero such that if $f_n(x_1) = f_n(x_2)$ for every n and $x_1, x_2 \notin Z$, then $x_1 = x_2$. Rényi [3] showed that maximality is sufficient to imply that the system

$$\{f_1^{m_1} \cdot f_2^{m_2} \cdot \ldots \cdot f_n^{m_n}\},$$

$m_i = 0, 1, 2, \cdots, n = 1, 2, 3, \cdots$, is closed in L^2. We [4] have shown that maximality is also necessary.

Clearly the sequence $\{\phi_n\}$ is maximal if and only if y is almost everywhere 1-1. We have further

LEMMA 4. *If* $\{\phi_n\}$ *is maximal, there is a metric automorphism* η *on* $(0, 1)$ *such that* $\eta(x) = y(x)$ *a.e.*

By a *metric automorphism* of a set we mean a 1-1 measure preserving mapping of the set onto itself.

Applying these results to W-systems we have

THEOREM 1. *If* $\{\psi_n\}$ *is a W-system, then the following conditions are equivalent:*

(i) $\{\psi_n\}$ *is complete.*

(ii) *There is a metric automorphism* η *of* $(0, 1)$ *such that* $\psi_n(x) = w_n \circ \eta(x)$ *a.e. for every n.*

We see then that for complete W-systems the study of the series $\sum c_n\psi_n$ can be replaced by the study of the Walsh series $\sum c_n w_n$. We note further that for every $f \in L^p$, $p \geqq 1$, $f \sim \sum c_n\psi_n$, we have $g = f \circ \eta^{-1} \in L^p$ and $g \sim \sum c_n w_n$. Thus most results of an almost everywhere nature concerning the Walsh-Fourier series of functions in L^p are extendable to W-Fourier series.

This is still the case for some types of results even if $\{\psi_n\}$ is not complete, for in this case, if $g \in L^p$, $p \geqq 1$, $g \sim \sum c_n w_n$, then $h = g \circ y \in L^p$ and $h \sim \sum c_n\psi_n$.

Since Stein [5] has shown that there is an L^1 function whose Walsh-Fourier series diverges a.e., we have

THEOREM 2. *For any W-system there is an integrable function whose W-Fourier series diverges a.e.*

If $f \in L^2$, $f \sim \sum c_n\psi_n$, then there is an $f^* \in L^2$, $f^* \sim \sum c_n w_n$. Since Billard [6] has solved Lusin's problem for Walsh-Fourier series we have

THEOREM 3. *The W-Fourier series of an L^2 function converges a.e.*

Our final result concerns a peculiar type of invariance under rearrangement. We state the results for Walsh series. We do not believe that this has been noted previously.

Suppose $\{r_{n_i}\}, i=1, 2, \cdots$, is a rearrangement of the Rademacher system. This induces a rearrangement of the Walsh system by means of the relation

$$w_{m_i} = r_{n_{i_1}} \cdot r_{n_{i_2}} \cdot \ldots \cdot r_{n_{i_k}}$$

where

$$m_i = 2^{n_{i_1}} + \cdots + 2^{n_{i_k}}.$$

Such rearrangements we call *coherent.* We have the following result.

THEOREM 4. *If $\{w_{m_n}\}$ is a coherent rearrangement of the Walsh system, then the almost everywhere convergence and summability behaviors of the series $\sum c_n w_n$ and $\sum c_n w_{m_n}$ are the same.*

REFERENCES

1. G. Alexits, *Convergence problems of orthogonal series*, Pergamon, Budapest, 1961.
2. A. Zygmund, *Trigonometric series*. Vol. I, Cambridge Univ. Press, New York, 1959.
3. A. Rényi, *On a conjecture of H. Steinhaus*, Ann. Polon. Math. **25** (1952), 279–287.
4. D. Waterman, *On a problem of Steinhaus*, submitted to Studia Sci. Math. Hungarica.
5. E. M. Stein, *On limits of sequences of operators*, Ann. of Math. (2) **74** (1964), 140–170.
6. P. Billard, *Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace $L^2(0,1)$*, Studia Math. **28** (1967), 363–388.

WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202