A MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEMS WITH FUNCTIONAL DIFFERENTIAL SYSTEMS[1]

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In this note we present a maximum principle in integral form for optimal control problems with delay-differential system equations which also contain delays in the control. Recent related results for particular cases of the systems discussed below may be found in [1], [5], and [6]. Vector matrix notation will be used and we shall not distinguish between a vector and its transpose.

Let α_0 and t_0 be fixed in R^1 with $-\infty<\alpha_0<t_0$, $I=[\alpha_0, a)$ be a bounded interval containing $[\alpha_0, t_0]$, and put $I'=(t_0, a)$. For x continuous on I and t in I', the notation $F(x(\cdot), t)$ will mean that F is a functional in x, depending on any or all of the values $x(\tau)$, $\alpha_0\leqq\tau\leqq t$. $\bar{\Phi}$ will denote the class of absolutely continuous $n-1$ vector functions on $[\alpha_0, t_0]$. Let Ω be a given convex subset of the class of all bounded Borel measurable functions u defined on I into R^r, and $\mathfrak{J}$ be a given C^1 manifold in R^{2n-1}. The problem considered is that of minimizing

$$J[\bar{\phi}, u, \bar{x}, t_1] = \int_{t_0}^{t_1} f^0(\bar{x}(\cdot), u(\cdot), t)dt$$

over $\bar{\Phi}\times\Omega\times C(I, R^{n-1})\times I'$ subject to

$$\text{(i)}\quad \dot{\bar{x}}(t) = \bar{f}(\bar{x}(\cdot), u(\cdot), t) \qquad \text{a.e. on } [t_0, t_1],$$
$$\bar{x}(t) = \bar{\phi}(t) \qquad \text{on } [\alpha_0, t_0],$$
$$\text{(ii)}\quad (\bar{x}(t_0), \bar{x}(t_1), t_1) \in \mathfrak{J}.$$

We assume that $f=(f^0, \bar{f})=(f^0, f^1, \cdots, f^{n-1})$ is an n-vector functional of the form

$$f^i(\bar{x}(\cdot), u(\cdot), t) = h^i(\bar{x}(\cdot), t) + \int_{\alpha_0}^{t} u(s)d_s\eta(t, s)g^i(\bar{x}(s), t) \tag{1}$$
$$\text{for } i = 0, 1, \cdots, n-1,$$

where the integral is a Lebesgue-Stieltjes integral. Each $h^i(\bar{x}(\cdot), t)$ is

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assumed C^1 in $\bar{x}$ and measurable in t, and each $g^i(\bar{y}, t)$ is C^1 in $(\bar{y}, t)$ on R^n. The $r\times 1$ vector function $\eta(t, s)$ is measurable in t, s, and of bounded variation in s on $[\alpha_0, t]$. It is also assumed that the variation of η is dominated by an $L_1(I')$ function m. That is, $V_{s=\alpha_0}^t \eta(t, s) \leqq m(t)$ for $t\in I'$. Finally, suppose that given $\overline{X}$ compact, $\overline{X}\subset R^{n-1}$, there exists an $\bar{m}$ in $L_1(I')$ such that $h=(h^0, h^1, \cdots, h^{n-1})$ satisfies $|h(\bar{x}(\cdot), t)| \leqq \bar{m}(t)$ and $|dh[\bar{x}(\cdot), t; \bar{\psi}]| \leqq \bar{m}(t)\|\bar{\psi}\|_t$ for any $\bar{\psi}\in C(I, R^{n-1})$ and $\bar{x}\in C(I, \overline{X})$, where $\|\bar{\psi}\|_t = \sup\{|\bar{\psi}(s)| : s\in[\alpha_0, t]\}$ and dh is the Fréchet derivative of h with respect to $\bar{x}$. ($|A|$ denotes the Euclidean norm of A.)

If $(\bar{\Phi}^*, u^*, \bar{x}^*, t_1^*)$ is a solution of the above problem, we define the $n\times n-1$ matrix function $\bar{\eta}^*$ for $t\in I'$, $s\in[\alpha_0, t]$ by

$$\bar{\eta}^*(t, s) = \bar{\eta}_1^*(t, s) + \bar{\eta}_2^*(t, s). \tag{2}$$

where $\bar{\eta}_1^*$ is such that

$$dh[\bar{x}^*(\cdot), t; \bar{\psi}] = \int_{\alpha_0}^{t} d_s\bar{\eta}_1^*(t, s)\bar{\psi}(s)$$

for $t\in I'$ and all $\bar{\psi}\in C([\alpha_0, t], R^{n-1})$, and

$$\begin{aligned} \bar{\eta}_2^*(t, s) &\equiv -\int_s^t u^*(\beta)d_\beta\eta(t, \beta)g_{\bar{y}}(\bar{x}^*(\beta), t) && s < t, \\ &\equiv 0 && s \geqq t. \end{aligned}$$

(The existence of $\bar{\eta}_1^*$ is guaranteed by the Riesz Theorem.) Then we have the following necessary conditions.

THEOREM. *Let $(\bar{\Phi}^*, u^*, \bar{x}^*, t_1^*)$ be a solution to the problem under the assumptions above. In addition, suppose that t_1^* is a Lebesgue point of $f(\bar{x}^*(\cdot), u^*(\cdot), t)$. Then there exists a nontrivial n-vector function $\lambda(t) = (\lambda^0(t), \bar{\lambda}(t))$ of bounded variation on $[t_0, t_1^*]$, continuous at t_1^*, satisfying*

(a) $\lambda^0(t) =$ *constant* $\leqq 0$, $\lambda(t_1^*) \neq 0$,

$$\bar{\lambda}(t) + \int_t^{t_1^*} \lambda(\beta)\bar{\eta}^*(\beta, t)d\beta = \bar{\lambda}(t_1^*) \qquad \text{for } t \in [t_0, t_1^*)$$

where $\bar{\eta}^*$ is defined by (2).

(b)
$$\int_{t_0}^{t_1^*} \lambda(t)f(\bar{x}^*(\cdot), u^*(\cdot), t)dt \geqq \int_{t_0}^{t_1^*} \lambda(t)f(\bar{x}^*(\cdot), u(\cdot), t)dt$$

for all $u\in\Omega$.

(c) *The* $2n-1$ *vector*

$$\left(-\bar{\lambda}(t_0) + \int_{t_0}^{t_1^*} \lambda(\beta)\{\bar{\eta}^*(\beta, \alpha_0) - \bar{\eta}^*(\beta, t_0)\}d\beta, \bar{\lambda}(t_1^*), -\lambda(t_1^*)\cdot f^*(t_1^*)\right)$$

is orthogonal to $\mathfrak{I}$ *at* $(\bar{x}^*(t_0), \bar{x}^*(t_1^*), t_1^*)$, *where* $f^*(t_1^*) \equiv f(\bar{x}^*(\cdot), u^*(\cdot), t_1^*)$.

The proof of this theorem involves showing that the class of functions $\mathfrak{F} = \{F(\bar{x}(\cdot), t): F(\bar{x}(\cdot), t) = f(\bar{x}(\cdot), u(\cdot), t), u \in \Omega\}$ is absolutely quasiconvex [2] and then using necessary conditions for extremals given in [2]. Absolute quasiconvexity is a generalization of ideas due to Gamkrelidze [4], who first obtained an integral maximum principle for control problems with ordinary differential system equations. The inequality in (b) is a maximum principle in integral form for the above described optimal control problem.

In many particular cases of the systems defined by (1), one can show that the multipliers $\bar{\lambda}$ are actually absolutely continuous and satisfy (a) in differentiated form. This differentiated form becomes the usual known multiplier equation for systems with simple time lags in the state variables (see [1]). The transversality conditions given in (c) also can be reduced to a simpler form for many special cases of (1).

Included in (1) are many integro-differential systems and time lag systems which appear in physical problems. For example, if one modifies slightly the biological population model formulated by Cooke in [3], one obtains the system equation

$$\dot{x}(t) = u(t-\tau)x(t-\tau) + \beta(t)u(t-\tau-\theta(t))x(t-\tau-\theta(t))$$

where $x(t)$ is the number in the population at time t, $u(t)$ is the conception rate at time t, and τ is the gestation period. Systems with

$$h(\bar{x}(\cdot), t) = \int_{\alpha_0}^{t} A(t, s)q(\bar{x}(s), t)ds,$$

which arise in the study of reactor dynamics [7], and with $h(\bar{x}(\cdot), t) = G(\bar{x}_t, t)$, where $\bar{x}_t = \bar{x}(t+\theta)$, $\theta \in [-T, 0]$, are also special cases of (1).

References

1. H. T. Banks, *Necessary conditions for control problems with variable time lags* J. SIAM Control **6** (1968), 9–47.

2. ———, *Variational problems involving functional differential equations*, J. SIAM Control **7** (1969).

3. K. L. Cooke, *Functional-differential equations: Some models and perturbation problems, differential equations and dynamical systems*, Academic Press, New York, 1967.

4. R. V. Gamkrelidze, *On some extremal problems in the theory of differential equations with applications to the theory of optimal control*, J. SIAM Control **3** (1965), 106–128.

5. E. B. Lee, *Geometric theory of linear controlled systems* (to appear).

6. ———, *Variational problems for systems having delay in the control action* (to appear).

7. J. J. Levin and J. A. Nohel, *A system of nonlinear integro-differential equations*, Michigan Math. J. **13** (1966), 257–270.

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