ON PROPERTIES OF SELF RECIPROCAL FUNCTIONS

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Following is the notation of Hardy and Titchmarsh [1]. We denote a function as $R\mu$ if it is self reciprocal for Hankel transforms of order $\mu$, so that it is given by the formula

\[(1.1) \quad f(x) = \int_0^\infty J_\mu(xy)f(y)\sqrt{xy}dy,\]

where $J_\mu(x)$ is a Bessel function of order $\mu$. For $\mu = \frac{1}{2}$ and $-\frac{1}{2}$, $f(x)$ is denoted as $R_s$ and $R_e$, respectively.

Brij Mohan [2] has shown that if $f(x)$ is $R\mu$, and

\[(1.2) \quad \theta(s) = \theta(1 - s) \quad \text{and} \quad 0 < c < 1,
\]

then $P(x)$ is a Kernel transforming $R\mu (R_s)$ into $R_v (R_e)$. As an example of (1.2) Brij Mohan has shown that the function

\[(1.3) \quad x^{s+1/2}e^{-x}
\]

is a Kernel transforming $R_s$ into $R_{s+1}$. In particular, putting $\nu = \frac{1}{2}$, we find that the Kernel

\[(1.4) \quad xe^{-x},
\]

transforms $R_s$ into $R_{3/2}$. Again, I have shown in a previous paper [3] that the Kernel

\[(1.5) \quad \sqrt{xe^{-x/2}},
\]

transforms $R_1$ into $R_2$. From (1.4) and (1.5) we find that “A Kernel transforming $R_1$ into $R_2$ will have its square transforming $R_s$ into $R_{3/2}$.” Again Sneddon [4] has shown that

\[\int_0^\infty e^{-x}x^m\ln(x)dx = (-1)^mm! \int_0^\infty \frac{d^{n-m}}{dn^{n-m}}(x^n e^{-x})dx,
\]

$\ln(x)$ being Laguerre polynomial of order $n$. Putting $m = n$, we obtain that
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\[
\int_0^\infty e^{-x^n} \ln(x) \, dx = (-1)^n n! \int_0^\infty x^n e^{-x} \, dx.
\]

On writing \(x^2\) for \(x\), we find that

\[
\int_0^\infty e^{-x^2} x^{2n+1} \ln(x^2) \, dx = (-1)^n n! \int_0^\infty x^{2n+1} e^{-x^2} \, dx.
\]

Brij Mohan [5] has shown that the function

\[
e^{-x^{2/2} x^{2n+1/2}}
\]

is \(R_{2n}\); while Howell [6] has shown that the function

\[
e^{-x^{2/2} x^{1/2} \ln(x^2)}
\]

is \(R_0\).

Hence, from (1.8) and (1.9) we find that the integral on the left-hand side of (1.7) is the product of the functions which are \(R_0\) and \(R_{2n}\). Again, by writing the integrand on the right-hand side of (1.7) in the form

\[
x^n e^{-x^{2/2} x^{n+1/2} e^{-x^2/2}},
\]

we find from (1.8) that it is a product of two functions which are \(R_{(n-1/2)}\) and \(R_{(n+1/2)}\). Applying the results of (1.8), (1.9) and (1.10) to (1.7), we conclude that the integral of a product of two functions which are \(R_0\) and \(R_{2n}\) is the integral of a product of two functions which are \(R_{n-1/2}\) and \(R_{n+1/2}\). This may be compared with a theorem given by me in a previous paper [7].

Again, the integrand on the right-hand side of (1.7) may also be written in the form

\[
(x^{n+1/2} e^{-x^2/2})^2,
\]

which is a square of an \(R_n\) function. Hence from (1.8), (1.9), (1.11) and (1.7), we further conclude that the integral of a square of an \(R_n\) function is the integral of the product of two functions which are \(R_0\) and \(R_{2n}\). My thanks are due to Dr. Brij Mohan for his constant guidance in my research work.

REFERENCES

REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON $L^p$ SPACES

BY V. J. MIZEL

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This note describes integral representations obtained for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(T)$ ($1 \leq p \leq \infty$) associated with an arbitrary $\sigma$-finite measure space $(T, \Sigma, \mu)$. The class of functionals considered here differs from those considered in [1], [3], [7], [8], [9] and its study is mainly motivated by its close connection with nonlinear integral equations [6].

In the study of nonlinear integral equations there is a fundamental class of nonlinear transformations, called Urysohn operators [6], taking measurable functions to measurable functions and having the form

\[(Ax)(s) = \int_T \phi(s; x(t), t)dt\]

where $S, T$ are Lebesgue measurable subsets of $\mathbb{R}^n$ and $\phi: S \times \mathbb{R} \times T \to \mathbb{R}$ is a real valued function which is measurable on $S \times T$ for each fixed value of its second argument and continuous on $\mathbb{R}$ for almost all arguments in $S \times T$. An important subclass of (1) consists of those Urysohn operators whose range is in $C(S)$ where $S$ is compact. This subclass includes the case in which the kernel $\phi$ is independent of its first argument, so that the operator reduces to a real valued functional:

\[F(x) = \int_T \phi(x(t), t)dt.\]

Functionals of the form (2) also play an important role in the theory of generalized random processes in probability [5].