ON PROPERTIES OF SELF RECIPROCAL FUNCTIONS

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Following is the notation of Hardy and Titchmarsh [1]. We denote a function as $R\mu$ if it is self reciprocal for Hankel transforms of order μ, so that it is given by the formula

$$f(x) = \int_0^\infty J_\mu(xy)f(y)\sqrt{xy}\,dy, \tag{1.1}$$

where $J_\mu(x)$ is a Bessel function of order μ. For $\mu=\frac{1}{2}$ and $-\frac{1}{2}$, $f(x)$ is denoted as R_s and R_c respectively.

Brij Mohan [2] has shown that if $f(x)$ is $R\mu$, and

$$P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s\Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\theta(s)x^{-s}ds,$$

where

$$\theta(s) = \theta(1-s) \quad \text{and} \quad 0 < c < 1, \tag{1.2}$$

then $P(x)$ is a Kernel transforming R_μ (R_ν) into R_ν (R_μ). As an example of (1.2) Brij Mohan has shown that the function

$$x^{\nu+1/2}e^{-x} \tag{1.3}$$

is a Kernel transforming R_ν into $R_{\nu+1}$. In particular, putting $\nu=\frac{1}{2}$, we find that the Kernel

$$xe^{-x}, \tag{1.4}$$

transforms R_s into $R_{3/2}$. Again, I have shown in a previous paper [3] that the Kernel

$$\sqrt{x}e^{-x/2}, \tag{1.5}$$

transforms R_1 into R_2. From (1.4) and (1.5) we find that "A Kernel transforming R_1 into R_2 will have its square transforming R_s into $R_{3/2}$." Again Sneddon [4] has shown that

$$\int_0^\infty e^{-x}x^m \mathrm{Ln}(x)dn = (-1)^m m!\int_0^\infty \frac{d^{n-m}}{dn^{n-m}}(x^n e^{-x})dx,$$

$\mathrm{Ln}(x)$ being Laguerre polynomial of order n. Putting $m=n$, we obtain that

$$\int_0^\infty e^{-x}x^n \operatorname{Ln}(x)dx = (-1)^n n! \int_0^\infty x^n e^{-x}dx.$$

On writing x^2 for x, we find that

$$\int_0^\infty e^{-x^2}x^{2n+1} \operatorname{Ln}(x^2)dx = (-1)^n n! \int_0^\infty x^{2n+1}e^{-x^2}dx. \tag{1.7}$$

Brij Mohan [5] has shown that the function

$$e^{-x^2/2}x^{2n+1/2} \tag{1.8}$$

is R_{2n}; while Howell [6] has shown that the function

$$e^{-x^2/2}x^{1/2} \operatorname{Ln}(x^2), \tag{1.9}$$

is R_0.

Hence, from (1.8) and (1.9) we find that the integral on the left-hand side of (1.7) is the product of the functions which are R_0 and R_{2n}. Again, by writing the integrand on the right-hand side of (1.7) in the form

$$x^n e^{-x^2/2} n^{n+1} e^{-x^2/2}, \tag{1.10}$$

we find from (1.8) that it is a product of two functions which are $R_{(n-1/2)}$ and $R_{(n+1/2)}$. Applying the results of (1.8), (1.9) and (1.10) to (1.7), we conclude that the integral of a product of two functions which are R_0 and R_{2n} = the integral of a product of two functions which are $R_{n-1/2}$ and $R_{n+1/2}$. This may be compared with a theorem given by me in a previous paper [7].

Again, the integrand on the right-hand side of (1.7) may also be written in the form

$$(x^{n+1/2}e^{-x^2/2})^2, \tag{1.11}$$

which is a square of an R_n function. Hence from (1.8), (1.9), (1.11) and (1.7), we further conclude that the integral of a square of an R_n function = the integral of the product of two functions which are R_0 and R_{2n}. My thanks are due to Dr. Brij Mohan for his constant guidance in my research work.

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REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON L^p SPACES

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This note describes integral representations obtained for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(T)$ $(1 \leqq p \leqq \infty)$ associated with an arbitrary σ-finite measure space (T, Σ, μ). The class of functionals considered here differs from those considered in [1], [3], [7], [8], [9] and its study is mainly motivated by its close connection with nonlinear integral equations [6].

In the study of nonlinear integral equations there is a fundamental class of nonlinear transformations, called Urysohn operators [6], taking measurable functions to measurable functions and having the form

$$(1) \qquad (Ax)(s) = \int_T \phi(s; x(t), t)dt$$

where S, T are Lebesgue measurable subsets of R^n and $\phi: S \times R \times T \to R$ is a real valued function which is measurable on $S \times T$ for each fixed value of its second argument and continuous on R for almost all arguments in $S \times T$. An important subclass of (1) consists of those Urysohn operators whose range is in $C(S)$ where S is compact. This subclass includes the case in which the kernel ϕ is independent of its first argument, so that the operator reduces to a real valued functional:

$$(2) \qquad F(x) = \int_T \phi(x(t), t)dt.$$

Functionals of the form (2) also play an important role in the theory of generalized random processes in probability [5].