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REPRESENTATION OF NONLINEAR TRANSFORMATIONS ON L^p SPACES

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This note describes integral representations obtained for a class of nonlinear functionals and nonlinear transformations on the spaces $L^p(T)$ ($1 \leq p \leq \infty$) associated with an arbitrary σ -finite measure space (T, Σ, μ) . The class of functionals considered here differs from those considered in [1], [3], [7], [8], [9] and its study is mainly motivated by its close connection with nonlinear integral equations [6].

In the study of nonlinear integral equations there is a fundamental class of nonlinear transformations, called Urysohn operators [6], taking measurable functions to measurable functions and having the form

$$(1) \quad (Ax)(s) = \int_T \phi(s; x(t), t) dt$$

where S, T are Lebesgue measurable subsets of R^n and $\phi: S \times R \times T \rightarrow R$ is a real valued function which is measurable on $S \times T$ for each fixed value of its second argument and continuous on R for almost all arguments in $S \times T$. An important subclass of (1) consists of those Urysohn operators whose range is in $C(S)$ where S is compact. This subclass includes the case in which the kernel ϕ is independent of its first argument, so that the operator reduces to a real valued functional:

$$(2) \quad F(x) = \int_T \phi(x(t), t) dt.$$

Functionals of the form (2) also play an important role in the theory of generalized random processes in probability [5].

Our main result gives an abstract characterization, for all σ -finite measure spaces $T = (T, \Sigma, \mu)$ and all compact Hausdorff spaces, of nonlinear transformations $A: L^p(T) \rightarrow C(S)$, $1 \leq p \leq \infty$, which have the form (1). In particular we characterize functionals on $L^p(T)$ having the form (2). This latter characterization extends earlier results [7], [8] involving functionals of the form

$$(3) \quad F(x) = \int_T \phi(x(t)) d\mu$$

defined on (essentially) nonatomic σ -finite measure spaces. Detailed proofs and related results will be given elsewhere.

Hereafter $T = (T, \Sigma, \mu)$ will denote a σ -finite measure space and $M(T)$ will denote the class of real valued measurable functions on T .

DEFINITION. A real valued function $\phi: R \times T \rightarrow R$ is said to be of *Caratheodory* type for T , denoted $\phi \in \text{Car}(T)$, if it satisfies

- (i) $\phi(\cdot, t): R \rightarrow R$ is continuous for almost all $t \in T$,
- (ii) $\phi(c, \cdot): T \rightarrow R$ is measurable for all $c \in R$.

Since for each simple function x , the function $\phi \circ x$ defined by $(\phi \circ x)(t) = \phi(x(t), t)$ is in $M(T)$, it follows by taking limits of sequences of simple functions and using (i) that for every $x \in M(T)$, $\phi \circ x$ is also in $M(T)$.

DEFINITION. Given a number p , $1 \leq p \leq \infty$, a function $\phi \in \text{Car}(T)$ is said to be in the *Caratheodory p -class* for T , denoted $\phi \in \text{Car}^p(T)$, if it satisfies

$$\phi \circ x \in L^1(T) \quad \text{for } x \in L^p(T).$$

[For the case of a finite nonatomic T it is known [6, p. 27] that ϕ is in $\text{Car}^p(T)$, $1 \leq p < \infty$, if and only if

$$|\phi(x, t)| \leq a(t) + b|x|^p \text{ for some } a \in L^1(T).]$$

THEOREM. Let $T = (T, \Sigma, \mu)$ be a finite measure space. Let F be a real valued functional on $L^p(T)$, $1 \leq p \leq \infty$, which satisfies

- (i) $F(x+y) - F(x) - F(y) = \text{const.} = c_F$ whenever $xy = 0$ a.e.,
- (ii) F is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$,
- (iii) F is continuous relative to L^p norm, if $p < \infty$, and is continuous with respect to bounded a.e. convergence, if $p = \infty$.

Then there exists a function $\phi \in \text{Car}^p(T)$ such that

$$(*) \quad F(x) = -c_F + \int_T \phi \circ x d\mu \quad \text{for } x \in L^p(T).$$

Moreover ϕ can be taken to satisfy

$$(a) \quad \phi(0, \cdot) = 0 \text{ a.e.}$$

and is then unique up to sets of the form $R \times N$ with N a null set in T .

Conversely, for every $\phi \in \text{Car}^p(T)$ satisfying (a), and for every $c_F \in R$, (*) defines a functional satisfying (i), (ii) and (iii).

The above result extends to σ -finite measure spaces. For $p = \infty$ it remains valid as is. For $p < \infty$ it is valid if the phrase 'bounded subset of $L^\infty(T)$ ' is replaced by 'bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.'

The proof occurs in two parts. First we consider the case $p = \infty$. The observation that $F_1 = F + c_F$ is a functional of the same type with $c_{F_1} = 0$ permits a reduction to the case $c_F = 0$. The construction of ϕ from F now depends on the fact that for each real number h the set function ν_h defined by

$$\nu_h(E) = F(h\chi_E),$$

where χ_E denotes the characteristic function of $E \in \Sigma$, is by (i) and (iii) a μ -continuous measure on T . Hence by the Radon-Nikodym theorem there exists for each h a density $\phi_h = \phi(h, \cdot)$ corresponding to ν_h . The proof that the family $\{\phi_h | h \in R\}$ defines a function $\phi \in \text{Car}^\infty(T)$ is a generalization of the classical proof of existence of a Lebesgue set for each function $f \in L^1(R)$. In fact the classical argument corresponds to the particular ϕ given by $\phi(x, t) = |x - f(t)|$. The validity of the representation (*) is then established by use of the Vitali convergence theorem.

The proof of the converse involves a modification of Nemytskii's argument for demonstrating that for any $\phi \in \text{Car}(T)$ the mapping $x \rightarrow \phi \circ x$ preserves convergence in measure [6], together with the Banach-Saks theorem.

For the case $p < \infty$ the argument is now based on the observation that $F' = F|_{L^\infty(T)}$ satisfies the hypotheses for the case $p = \infty$ and therefore possesses a unique representing function ϕ' satisfying (a). Vitali's convergence theorem then implies that $\phi = \phi'$ has the given properties. The converse utilizes a result of Krasnoselskii's on continuity of the transformation $x \rightarrow \phi \circ x$.

REMARK. In the linear case this reduces to the Riesz representation theorem, modulo a proof that $\phi(x, t) = xa(t)$ is in $\text{Car}^p(T)$ if and only if $a \in L^q(T)$, $1 \leq p < \infty$.

THEOREM. *With (T, Σ, μ) as in the preceding theorem let A be a transformation such that $A: L^p(T) \rightarrow C(S)$, $1 \leq p \leq \infty$, where S is a compact Hausdorff space. Suppose A satisfies the conditions*

- (i_A) $A(x+y) = A(x) + A(y)$ when $xy = 0$ a.e.,
 (ii_A) A is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$,
 (iii_A) A is continuous relative to L^p norm, if $p < \infty$, and is continuous with respect to bounded a.e. convergence, if $p = \infty$.

Then there exists a transformation $\Phi: S \rightarrow \text{Car}^p(T)$ such that

$$(*) \quad A(x)(s) = \int_T \Phi(s) \circ x \, d\mu.$$

The transformation Φ can be taken to satisfy

- (a) $\Phi(s) \circ 0 = 0$ a.e. for all $s \in S$,
 in which case $\Phi(s)$ is unique, for each s , up to sets of the form $R \times N$ with N a null set in T . Moreover Φ has the following additional properties:
 (b) the mapping $s \rightarrow \Phi(s) \circ x \in L^1(T)$ is weakly continuous for each $x \in L^p(T)$,
 (c) the mapping $x \rightarrow \Phi(s) \circ x$ is uniformly continuous relative to L^∞ norm on each bounded subset of $L^\infty(T)$, uniformly in s ,
 (d) the mapping $x \rightarrow \Phi(s) \circ x$ is weakly continuous, on $L^p(T)$, uniformly in s , if $p < \infty$; if $x_n \rightarrow x$ boundedly a.e. then $\lim_{\mu(E) \rightarrow 0} \int_E (\Phi(s) \circ x_n) \, d\mu \rightarrow 0$, uniformly in s and n , if $p = \infty$.

Conversely, every transformation $\Phi: S \rightarrow \text{Car}^p(T)$, $1 \leq p \leq \infty$, satisfying (a), (b), (c), (d) determines by means of (*) a transformation $A: L^p(T) \rightarrow C(S)$ satisfying (i_A), (ii_A) and (iii_A).

The above result also extends to σ -finite measure spaces. For $p = \infty$ it is valid if the following condition is added:

- (e) if $x_n \rightarrow x$ boundedly a.e., then for any sequence $E_j \downarrow \emptyset$, $\int_{E_j} (\Phi(s) \circ x_n) \, d\mu \rightarrow 0$, uniformly in s and n . For $p < \infty$ it is valid if the phrase 'bounded subset of $L^\infty(T)$ ' is replaced by 'bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.'

The proof utilizes the preceding theorem on functionals together with the Vitali-Hahn-Saks theorem on convergence of measures.

REMARK. For the linear case this result is well known (see [4, p. 490]).

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